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WATER WAVES. V.

by

John V. Wehausen

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F. EXACT SOLUTIONS

The word "exact" in this context is generally understood to mean solutions in which there has been no approximation in the equations or boundary conditions. However, this usage of the word does not exclude neglect of viscosity and, in fact, since positive results have been obtained only for perfect fluids, the treatment below will be restricted to them. Indeed, the present results in the theory of exact solutions are restricted, with few exceptions, to a very special class of motions, namely, those which can be represented as steady two-dimensional flows.

In section 32 some general theorems will be established. In section 33 waves of maximum amplitude-to-length ratio are discussed; because the methods are intimately related, we have also included in this section a discussion of Havelock's method of approximating periodic waves. Section 34 treats methods of obtaining explicit exact solutions and of various ones which have been obtained. In section 35, the last, existence theorems are discussed, but only in a descriptive way, for proofs are highly technical and lengthy.

32. Some general theorems

This section will be devoted to several theorems of a rather general nature concerning the motion of a fluid with free surface in a gravitational field. The theorems in 32α are mostly of a kinematical nature and are associated with the phenomenon of mass transport already discussed in section 27α. The last part of this section is devoted to several theorems on energy and momentum. In section 32β some theorems concerning waves in heterogeneous fluids will be established. In 33γ several different ways of formulating the problem of motion with a free surface will be described.

32α. Kinematical theorems--mass transport--energy integrals

The first theorem, due to M. S. Longuet-Higgins [1953], is independent of the presence of a free surface or of the nature of the time dependence. Let $f(z) = \varphi + i\psi$ describe a space-periodic motion, i.e. $f(z + n\lambda) = f(z)$. The definition of φ will be normalized so that

$$\int_0^\lambda \varphi(x, y, t)\, dx = 0 \tag{32.1}$$

Note that if this condition holds for one value of y, it holds for all since

$$\frac{\partial}{\partial y}\int_0^\lambda \varphi\, dx = \int_0^\lambda \varphi_y\, dx = -\int_0^\lambda \psi_x\, dx = \psi(\lambda, y, t) + \psi(0, y, t) = 0.$$

In equation (2.10') we shall write

$$\frac{p}{\rho} = \frac{p_0}{\rho} - gy + \frac{p_d}{\rho}, \quad \frac{p_d}{\rho} = A(t) - \varphi_t - \frac{1}{2}(u^2 + v^2). \tag{32.2}$$

In the following we define an average by

$$\bar{F}(y,t) = \frac{1}{\lambda}\int_0^\lambda F(x,y,t)\,dx. \qquad (32.3)$$

<u>Theorem</u>: In a non-uniform space-periodic motion $\overline{u^2}, \overline{v^2}, -\overline{p_d}$ each decrease with increasing depth , provided either $\varphi_y(x,-h,t)=0$ or $\lim_{y\to-\infty} \varphi_y = 0$.

This may be proved as follows. Consider first $q^2 = u^2+v^2$. Then

$$\begin{aligned}\frac{\partial}{\partial y}\overline{q^2} &= \frac{\partial}{\partial y}\frac{1}{\lambda}\int_0^\lambda (\varphi_x^2+\varphi_y^2)\,dx = \frac{2}{\lambda}\int_0^\lambda (\varphi_x\varphi_{xy}+\varphi_y\varphi_{yy})\,dx \\ &= \frac{2}{\lambda}\int_0^\lambda \left[(\varphi_y\varphi_x)_x - 2\varphi_y\varphi_{xx}\right]dx \\ &= \frac{2}{\lambda}\left[\varphi_y\varphi_x\right]_0^\lambda - \frac{4}{\lambda}\int_0^\lambda \varphi_y\varphi_{xx}\,dx \\ &= -\frac{4}{\lambda}\int_0^\lambda \varphi_y\varphi_{xx}\,dx. \end{aligned} \qquad (32.4)$$

By a similar computation it follows that

$$\frac{\partial^2}{\partial y^2}\overline{q^2} = \frac{4}{\lambda}\int_0^\lambda (\varphi_{xx}^2+\varphi_{xy}^2)\,dx > 0, \qquad (32.5)$$

since we have assumed that φ_x is not constant. It is evident from (32.4) that, if the fluid is bounded below by $y=-h$, then

$$\frac{\partial}{\partial y}\overline{q^2}(-h,t) = 0; \qquad (32.6)$$

if it is infinitely deep, it is an assumed boundary condition that $\varphi_y \to 0$ as $y\to-\infty$ and hence

$$\frac{\partial}{\partial y}\overline{q^2} \to 0 \quad \text{as} \quad y \to -\infty. \tag{32.7}$$

In either case it then follows from (32.5) that $\partial\overline{q^2}/\partial y$ is an increasing function of y and hence

$$\frac{\partial}{\partial y}\overline{q^2} \geq 0, \tag{32.8}$$

with equality occurring only for $y=-h$. In fact, even more can be concluded, for (32.5) is like $\overline{q^2}$ itself with φ replaced by $2\varphi_x$. Hence, by repeating the above reasoning one may establish that

$$\frac{\partial^{2n}}{\partial y^{2n}}\overline{q^2} > 0, \quad \frac{\partial^{2n-1}}{\partial y^{2n-1}}\overline{q^2} \geq 0, \quad n=1,2,\cdots. \tag{32.9}$$

Next consider $\overline{u^2}-\overline{v^2}$. A similar computation shows that

$$\frac{\partial}{\partial y}(\overline{u^2}-\overline{v^2}) = \frac{2}{\lambda}\int_0^\lambda (\varphi_x\varphi_y)_x\,dx = \frac{2}{\lambda}\left[\varphi_x\varphi_y\right]_0^\lambda = 0.$$

Hence

$$\overline{u^2}-\overline{v^2} = C(t) = \overline{u^2}\Big|_{y=-h \text{ or } -\infty}. \tag{32.10}$$

It follows from (32.8) that

$$\overline{u^2} = \tfrac{1}{2}\left[\overline{q^2}+C\right], \quad \overline{v^2} = \tfrac{1}{2}\left[\overline{q^2}-C\right], \quad -\overline{p_d} = \tfrac{1}{2}\overline{q^2}-A(t) \tag{32.11}$$

are each increasing functions of y, i.e., they decrease with increasing depth. For infinite depth Longuet-Higgins shows further that, if axes are chosen such that $u=0$ at $y=-\infty$, then

the quantities $|u|$, $|v|$ and $|p_d|$ all decrease exponentially to zero. He had shown earlier [1950] for exact waves (we shall not carry through the proof) that

$$\overline{p_d} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{\eta^2} - \overline{v^2}. \tag{32.12}$$

Hence it follows that

$$\overline{p_d}\Big|_{y=-h \text{ or } -\infty} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{\eta^2}. \tag{32.13}$$

For purely progressive waves this quantity vanishes, but we recall that for standing waves we found earlier a constant pressure fluctuation of double the wave frequency [see (27.62) and (27.65)] if second-order terms were retained.

Mass transport. In section 27 [see (27.39) and (27.41)] it was shown that a forward drift, called "mass transport", occurred in progressive waves if second-order terms were taken into account. It was shown by Rayleigh [1876] in a proof valid only for infinitely deep fluid that mass transport must always occur. The proof is independent of the dynamical free-surface condition. Levi-Civita [1912] and later Ursell [1953] developed methods of analysis to include both finite depth and nonperiodic waves; essentially Ursell's analysis has also been given by Nekrasov [1951] for infinite depth. The analysis given below is due to Longuet-Higgins [1953] and is similar to that used in the preceding theorem. We note that Starr [1945] has also given an instructive and perspicuous derivation of Rayleigh's theorem for infinite depth.

Take the wave as moving to the left with velocity c (in the sense of section 7) and impose a uniform velocity c in the opposite direction, so that the motion is reduced to a steady one, generally in the positive x-direction in the sense that $u > \varepsilon > 0$. We may then write the complex potential in the form

$$f(z) = \Phi + i\Psi = cz + \varphi + i\psi, \tag{32.14}$$

where $Re\, f' > \varepsilon > 0$ and

$$\Phi(x+n\lambda, y) = nc\lambda + \Phi(x,y) \quad , \quad \Psi(x+n\lambda, y) = \Psi(x,y). \tag{32.15}$$

We take $\Phi = 0$ at a crest and assume $\Psi = 0$ as the free-surface streamline and $\Psi = -Q$ as the bottom streamline if the depth is finite. One may invert the relation $f = f(z)$ and obtain $z = z(f)$. Then, since $q^2 \neq 0$,

$$z'(f) = \frac{1}{f'(z)} = \frac{\Phi_x + i\,\Phi_y}{\Phi_x^2 + \Phi_y^2} = \frac{1}{q^2}(u + iv) = x_\Phi + i\,y_\Phi \tag{32.16}$$

Denote by $T(\Psi)$ the time required for a given particle to progress one wave length along a streamline Ψ = const. In the original wave motion, the time elapsed between the passage of two successive crests over a given point is λ/c. If $T > \lambda/c$, the particle is being transported with the wave and it will be reasonable to call

$$U(\Psi) = c - \frac{\lambda}{T(\Psi)} \tag{32.17}$$

the mass transport in the direction of wave motion. The following theorem is true.

Theorem: Both T and U decrease with increasing depth, and, with the assumed definition of c, $U > 0$.

The theorem may be proved by the following computation:

$$T(\psi) = \int_0^{s(\lambda)} \frac{1}{q} ds = \int_0^{c\lambda} \frac{1}{q} \frac{\partial s}{\partial \phi} d\phi = \int_0^{c\lambda} (x_\phi^2 + y_\phi^2) d\phi$$

$$= \int_0^{c\lambda} (x_\phi^2 + x_\psi^2) d\phi, \tag{32.18}$$

$$T'(\psi) = 4 \int_0^{c\lambda} x_\phi x_{\phi\psi} d\phi, \tag{32.19}$$

$$T''(\psi) = 4 \int_0^{c\lambda} (x_{\psi\phi}^2 + x_{\psi\psi}^2) d\phi. \tag{32.20}$$

The details of the computation are almost identical with those used in deriving (32.4) and (32.5). Since

$$x_\psi = -y_\phi = -\frac{1}{q^2}\phi_y = 0 \quad \text{on } \psi = -Q,$$

it follows from (32.19) that $T'(-Q) = 0$. Then, since $T''(\psi) > 0$ unless the flow is uniform, it follows that

$$T'(\psi) \geq 0, \tag{32.21}$$

with equality holding only for $\psi = -Q$. As in the earlier theorem, the computations can be exterded to yield

$$T^{(2n)}(\psi) > 0, \quad T^{(2n-1)}(\psi) \geq 0. \tag{32.22}$$

It now follows immediately from (32.17) that

$$U'(\psi) \geq 0, \tag{32.23}$$

with the equality holding only for the bottom streamline. If the fluid is infinitely deep, then $U' > 0$ for all ψ. To complete the proof we must show that $U > 0$. If the fluid is infinitely deep, it is evident that

$$\lim_{\psi \to -\infty} T(\psi) = \frac{\lambda}{c}. \tag{32.24}$$

Hence $\lim U = 0$ as $\psi \to -\infty$ and the conclusion follows from $U' > 0$. If the depth is finite, we compute

$$T(-Q) = \int_0^{c\lambda} x_\phi^2 \, d\phi > \frac{1}{c\lambda}\left[\int_0^{c\lambda} x_\phi \, d\phi\right]^2 = \frac{1}{c\lambda}\lambda^2 = \frac{\lambda}{c}; \tag{32.25}$$

here use has been made of the Schwarz-Bunyakovskii inequality. (We have written $>$ rather than $\geqslant$, for the equal sign will hold only in the trivial case of a uniform flow.) It now follows that $U(-Q) > 0$ and hence that

$$U(\psi) > 0 \tag{32.26}$$

since $U' \geqslant 0$. This completes the proof of the theorem.

The method of analysis can be extended to prove an analogous theorem for nonperiodic steady motions which approach uniform flows as $x \to \pm\infty$, in particular, to the solitary wave.

<u>Momentum and energy integrals.</u> We close this section with several momentum and energy integrals, most of which have been found by Levi-Civita [1912, 1921], Starr [1947ab, 1948] and Starr and Platzman [1948].

Let us again take the wave as moving to the left without change of form and impose an opposite velocity c which brings

the profile to rest (or, equivalently, consider the motion relative to a coordinate system moving with the wave). Let the velocity potential be as in (32.14). Consider the area bounded by two streamlines $\psi = \psi_1$ and $\psi = \psi_2$, say $y = \eta_1(x)$ and $y = \eta_2(x)$ and two vertical lines a wave length λ apart. To this area apply the theorem

$$\iint (\phi_x^2 + \phi_y^2)\, d\sigma = \oint \phi\, \phi_n\, ds. \tag{32.27}$$

This yields

$$\iint [(c+u)^2 + v^2]\, d\sigma = \int_{\eta_1(x_0+\lambda)}^{\eta_2(x_0+\lambda)} \phi(x_0+\lambda, y)\phi_x\, dy - \int_{\eta_1(x_0)}^{\eta_2(x_0)} \phi(x_0, y)\phi_x\, dy \tag{32.28}$$

since $\phi_n = 0$ on the streamlines. Moreover, since $\phi(x+\lambda, y) = c\lambda + \phi(x,y)$, $\phi_x(x+\lambda, y) = \phi_x(x,y) = c+u$ and $\eta_i(x+\lambda) = \eta_i(x)$, the right-hand side of (32.28) may be written as

$$c^2\lambda[\eta_2(x_0) - \eta_1(x_0)] + c\lambda \int_{\eta_1(x_0)}^{\eta_2(x_0)} \phi_x(x_0, y)\, dy$$

$$= c^2\lambda[\eta_2(x) - \eta_1(x)] + c\lambda \int_{\eta_1(x)}^{\eta_2(x)} u\, dy. \tag{32.29}$$

Expanding $(c+u)^2$ and rearranging give

$$\iint (u^2+v^2)\, d\sigma + 2c \iint u\, d\sigma + c^2 \iint d\sigma$$

$$= c^2\lambda[\eta_2(x) - \eta_1(x)] + c\lambda \int_{\eta_1(x)}^{\eta_2(x)} u\, dy. \tag{32.30}$$

If one now applies the operator $\lambda^{-1}\int_0^\lambda \cdots dx$ to (32.30), one obtains

$$\iint (u^2+v^2)\, d\sigma + c \iint u\, d\sigma = 0 \tag{32.31}$$

or, after multiplying by $\frac{1}{2}\rho$ and rearranging,

$$\iint \frac{1}{2}\rho(u^2+v^2)\,d\sigma = \frac{1}{2}c\iint -\rho u\,d\sigma, \tag{32.32}$$

i.e., the kinetic energy per wave length between two streamlines equals $\frac{1}{2}c$ times the momentum in the direction of the wave (here to the left).

Next let us write the integral (2.10') in the form

$$\frac{1}{2}\rho\left[(c+u)^2+v^2\right] + \rho g y + p = \frac{1}{2}\rho c_1^2, \tag{32.33}$$

the form of the constant having been chosen for later convenience. Write the terms $p+\rho g y$ as follows:

$$p+\rho g y = \frac{\partial}{\partial y}\left[y(p+\rho g y)\right] - y\frac{\partial}{\partial y}(p+\rho g y) = \frac{\partial}{\partial y}\left[y(p+\rho g y)\right] + y\frac{D}{Dt}v$$

$$= \frac{\partial}{\partial y}\left[y(p+\rho g y)\right] - v^2 + \frac{D}{Dt}(yv). \tag{32.34}$$

Here we have used the second equation of (2.6). We may now write (32.33) as follows

$$\frac{1}{2}\rho(u^2-v^2)+\rho c u + \frac{\partial}{\partial y}\left[y(p+\rho g y)\right] + \frac{D}{Dt}(yv) = \frac{1}{2}\rho(c_1^2-c^2). \tag{32.35}$$

Next let us integrate equation (32.35) over the same area as is described in the preceding paragraph. First consider $D(yv)/Dt$. Since the motion is steady in the selected coordinate system,

$$\frac{D}{Dt}(yv) = (u+c)\frac{\partial(yv)}{\partial x} + v\frac{\partial(yv)}{\partial y} = \frac{\partial}{\partial x}(u+c)yv + \frac{\partial}{\partial y}yv^2,$$

where the last equality follows from the continuity equation. Hence

$$\iint \frac{D}{Dt}(\gamma v)\,d\sigma = \oint \gamma v(u+c, v)\cdot \underset{=}{n}\,ds = \oint \gamma v \phi_n\,ds = 0 \tag{32.36}$$

since $\phi_n = 0$ on the streamline boundaries and the integrals over the vertical boundaries cancel from periodicity. The integrated equation then becomes

$$\iint \tfrac{1}{2}\rho(u^2-v^2)\,d\sigma + c\iint \rho u\,d\sigma + \int_0^\lambda \{\eta_2(x)[p(x,\eta_2)+\rho g\eta_2] - \eta_1(x)[p(x,\eta_1)+\rho g\eta_1]\}\,dx$$

$$= \tfrac{1}{2}\rho(c_1^2-c^2)\iint d\sigma. \tag{32.37}$$

If one eliminates the second integral by means of (32.32), one obtains

$$\iint \tfrac{1}{2}\rho u^2\,d\sigma + 3\iint \tfrac{1}{2}\rho v^2\,d\sigma - \int_0^\lambda \{\eta_2[p(x,\eta_2)+\rho g\eta_2] - \eta_1[p(x,\eta_1)+\rho g\eta_1]\}\,dx$$

$$= \tfrac{1}{2}\rho(c^2-c_1^2)\iint d\sigma. \tag{32.38}$$

Equation (32.38) has a simpler aspect if the two streamlines are taken as the free surface $\eta(x)$ and the bottom $y=-h$. Then $p(x,\eta(x)) = 0$ and the third integral becomes

$$\int_0^\lambda \rho g\eta_2^2(x)\,dx + h\int_0^\lambda [p(x,-h)-\rho g h]\,dx.$$

Moreover,

$$\int_0^\lambda [p(x,-h)-\rho g h]\,dx = 0 \tag{32.39}$$

if the x-axis is taken at the mean water level. This follows

from the following sequence of equations, similar to those used in (32.36):

$$\int_0^\lambda [p(x,-h) - \rho g h]\,dx = \iint \frac{\partial}{\partial y}[p(x,y) + \rho g y]\,d\sigma$$
$$- \iint \left[(u+c)\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v\right] d\sigma = -\iint \left[\frac{\partial}{\partial x}v(u+c) + \frac{\partial}{\partial y}v^2\right] d\sigma$$
$$= \oint v(u+c, v)\cdot \underline{n}\,d\sigma = -\oint v\phi_n\,d\sigma = 0. \qquad (32.40)$$

Equation (32.39) now allows us to give a simple physical interpretation of the constant c_1 in (32.33). For if (32.33) is integrated along $y = -h$, and account is taken of (32.39), one finds

$$\frac{1}{\lambda}\int_0^\lambda (u+c)^2\,dx = c_1^2 \geq c^2, \qquad (32.41)$$

i.e., c_1^2 is the mean square velocity of fluid along the bottom. The inequality follows easily from

$$\int_0^\lambda u(x,-h)\,dx = \int_0^\lambda \varphi_x(x,-h)\,dx = \varphi(\lambda,-h) - \varphi(0,-h) = 0. \qquad (32.42)$$

If the fluid is infinitely deep, $u \to 0$ as $y \to -\infty$, and (32.41) reduces to

$$c^2 = c_1^2. \qquad (32.43)$$

If, following (15.27), we let T_{av}, T_{xav}, T_{yav}, V_{av}, $\mathcal{M}_{av}$ denote the average kinetic energy, the contributions to this due to the x and y velocity components, potential energy, and momentum in the direction of wave motion, respectively, then

(32.32) and (32.88) may be expressed as follows:

$$2\mathcal{T}_{av} = c\mathcal{M}_{av}, \quad \mathcal{T}_{x\,av} + 3\mathcal{T}_{y\,av} = 2\mathcal{V}_{av} - \tfrac{1}{2}\rho(c_1^2 - c^2)h, \tag{32.44}$$

where the last term of the second equation is 0 for $h = \infty$. The first equation is essentially due to Levi-Civita [1912, 1921], the second to Starr [1947b].

We note another simple consequence of (32.41), due to Levi-Civita [1924]. Let us integrate (32.33) along the free surface for a wavelength and divide by $\frac{1}{2}\rho\lambda$. Then, remembering our choice of x-axis as the mean water level, we find

$$\frac{1}{\lambda}\int_0^\lambda [(c+u)^2 + v^2]\,dx = c_1^2. \tag{32.45}$$

On the other hand, if we compute the velocity at the intersection of the mean water level and the profile, we also find

$$(c+u)^2 + v^2 \big|_{y=0} = c_1^2. \tag{32.46}$$

Hence the absolute value of the velocity at the mean water level equals the root-mean-square velocity along the surface profile or along the bottom, or, indeed, along any streamline, for in the reasoning in (32.40) we could have substituted any streamline $y = \eta_1(x)$ for $y = -h$ and obtained

$$\int_0^\lambda [p(x, \eta_1(x)) - \rho g \eta_1(x)]\,dx = 0. \tag{32.47}$$

Starr and Platzman [1948] have used the relations above to derive some general relations concerning the flow of energy in a periodic wave. We recall that the average flux of energy in the

direction of wave motion is given by [cf. §8, (15.23) and (15.27)]

$$\mathcal{F}_{av} = \frac{1}{\lambda}\int_0^{\lambda} dx \int_{-h}^{\eta(x)} \rho c \varphi_x^2(x,y)\, dy = 2\, T_{x\,av}. \tag{32.48}$$

It follows from the second formula in (32.44) that

$$2\, T_{x\,av} = 3\, T_{av} - 2\, V_{av} + \frac{1}{2}\rho(c_1^2 - c^2)h \tag{32.49}$$

Hence, with $\mathcal{E}_{av} = T_{av} + V_{av}$, we obtain from (32.48)

$$\frac{\mathcal{F}_{av}}{\mathcal{E}_{av}} = \frac{1}{2} + \frac{5}{2}\,\frac{T_{av} - V_{av} + \frac{1}{5}\rho(c_1^2 - c^2)h}{\mathcal{E}_{av}}. \tag{32.50}$$

This should be compared with the result derived in section 15 for infinitesimal waves with neglect of surface tension [cf. (15.25) and (15.26)], namely, $\mathcal{F}_{av} = \frac{1}{2}\mathcal{E}_{av}$. Equation (32.50) is consistent with this, for to the order of approximation involved, $T_{av} = V_{av}$ and $c_1^2 = c^2$. However, for waves of finite height it was shown in section 27α [cf. eqs. (27.42), (27.43)] that to the second order of approximation $T_{av} > V_{av}$. Platzman [1947] has verified that this remains true when 4th-order terms are kept.

Several of the above results have analogues for steady motion of nonperiodic waves, provided that $\eta(x) \to 0$ as $x \to \pm\infty$ in such a way that $\int_{-\infty}^{\infty} \eta\, dx$ is finite. Under such circumstances $c_1^2 = c^2$ and the following results may be established (the notation is that of (15.31) with obvious extensions):

$$\mathcal{M}_{total} = c\int_{-\infty}^{\infty} \eta\, dx = c\,\mathcal{Q}_{total},$$

$$T_{x\,total} - T_{y\,total} = V_{total}, \tag{32.51}$$

$$\mathcal{T}_{x\,\text{total}} - \mathcal{T}_{y\,\text{total}} + 2V_{\text{total}} + (gh - c^2)\,\mathcal{A}_{\text{total}} = 0.$$

For details of the proof one may refer to Starr [1947b]. From the last two equations follows

$$c^2 = gh + 3\,V_{\text{total}}/\mathcal{A}_{\text{total}} > gh. \tag{32.52}$$

We note that the second equation of (32.51) is a special case of a more general one applying to any steady motion:

$$\mathcal{T}_x(x) - \mathcal{T}_y(x) - \mathcal{V}(x) = \text{const.} \tag{32.53}$$

where the constant is zero under the conditions of (32.51). The proof is analogous to that of (8.6). Here

$$\mathcal{T}_x(x) = \int_{-h}^{\eta(x)} \tfrac{1}{2}\rho u^2(x,y)\,dy, \quad \text{etc.}$$

32β. Waves in heterogeneous fluids

The first two theorems proved below are also true for homogeneous fluids and were first proved for this case. The last theorems deal specifically with heterogeneous fluids. In the extended form they are all due to Dubreil-Jacotin [1932].

A flow will be called barotropic if both the pressure and density are constant along streamlines. We first derive the energy integral for such flows. The equations (2.6) may be written in the following form in two dimensions:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial E}{\partial x} - v\zeta + \frac{\partial u}{\partial t}, \quad -\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial E}{\partial y} + u\zeta + \frac{\partial v}{\partial t}, \tag{32.54}$$

where

$$E = gy + \tfrac{1}{2}(u^2+v^2) \quad , \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \, .$$

Since ρ is assumed constant on a streamline, $u\rho_x + v\rho_y = 0$: it follows from (32.27) and the definition of E that

$$0 = u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y} + \frac{1}{2}\frac{\partial}{\partial t}\tilde{q}^2 = gv + \frac{1}{2}\frac{D}{Dt}\tilde{q}^2 = \frac{D}{Dt}E \qquad (32.55)$$

In particular, if the flow is steady, E is also constant along a streamline. For steady flow it is a consequence of the incompressibility condition that ρ is also constant along a streamline.

The following theorem was proved by Burnside [1915] for a homogeneous fluid. He gives two proofs, of which the second can be carried over to the present more general situation with no change. It will perhaps give more substance to the theorem if we remark that Gerstner's wave (see section 34β), which is not irrotational, satisfies the other conditions of the theorem.

Theorem: The only steady two-dimensional irrotational motion of a fluid subject to gravity for which all streamlines are also lines of constant pressure is a uniform flow.

Let the streamlines be given by $\psi(x,y) =$ const. Since, from the remark following (32.55), E = const. along a streamline, we may write

$$\frac{1}{2}(\psi_x^2 + \psi_y^2) + gy = E(\psi). \qquad (32.56)$$

(Burnside shows that one may generalize (32.56) by replacing gy by a function $g(y)$.) Since the motion is irrotational, $\Delta\psi = 0$

and hence also

$$\Delta \log(\psi_x^2 + \psi_y^2) = 0.$$

But then

$$\Delta \log[E(\psi) - gy] = 0,$$

which yields after some computation

$$2yE'(\psi)\psi_y = 2(E - gy)[E'^2 - (E - gy)E''] + g^2. \qquad (32.57)$$

We write this in the form

$$\psi_y(x, y) = G(\psi, y). \qquad (32.58)$$

It then follows from (32.56) and (32.58) that

$$\psi_{xx} = E' - GG_\psi, \quad \psi_{yy} = G_\psi \psi_y + G_y$$

or

$$E'(\psi) + G_y(\psi, y) = 0.$$

But then

$$\psi_y = -yE'(\psi) + \text{const}$$

and ψ is a function of y only. Hence, since $\Delta\psi = \psi_{yy} = 0$, ψ_y is a constant and the flow is uniform.

The next theorem was first proved by Levi-Civita [1925] for homogeneous fluids. Fenchel [1931] showed that his hypotheses could be weakened and Dubreil-Jacotin [1932] extended Fenchel's proof to heterogeneous fluids. The gist of the theorem is that if the surface profile moves without change of form, then the

whole velocity field is steady in a coordinate system moving with the surface. The theorem will be formulated in the moving coordinate system.

Theorem: Let a possibly heterogeneous fluid, bounded below by a horizontal plane $y = -h$, be flowing irrotationally in the x-direction with discharge rate $Q(t)$ and with a fixed surface profile $y = \eta(x)$. If η and u satisfy the conditions

$$-h < b_1 < \eta < b_2 , \qquad u > \varepsilon > 0, \tag{32.59}$$

then the velocity potential $f(z)$ is independent of t.

First we derive a boundary condition at the free surface. From the condition of constant pressure and the assumption that the surface profile is an invariant streamline it follows that

$$u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} = 0 \quad \text{on} \quad \psi = 0 .$$

It then follows as in (32.55) that

$$\frac{DE}{Dt} = gv + \frac{Dq^2}{Dt} = 0 \quad \text{on} \quad \psi = 0 . \tag{32.60}$$

However, this conclusion holds now only on this one streamline.

The complex potential $f(z,t) = \phi + i\psi$ maps the region of the z-plane occupied by fluid onto the strip $-Q(t) \le \psi \le 0$, where the free surface corresponds to $\psi = 0$, the bottom to $\psi = -Q$ and $x = \pm\infty$ to $\phi = \pm\infty$. Let $F(z) = \Phi + i\Psi$ be the mapping, unique up to an additive real constant, of the fluid region onto the strip $-1 \le \Psi \le 0$ with $x = \pm\infty$ corresponding to $\Phi = \pm\infty$. Then

$$f(z,t) = Q(t) F(z) \tag{32.61}$$

evidently satisfies the requirements for $f(z,t)$ and, in fact, is determined uniquely, up to the added constant in F, by $Q(t)$ and $\eta(x)$. Now substitute $\phi(x,y,t)=Q(t)\Phi(x,y)$ into (32.60):

$$g Q \Phi_y + Q Q'[\Phi_x^2 + \Phi_y^2] + Q^2[\Phi_{xx}\Phi_x^2 + 2\Phi_{xy}\Phi_x\Phi_y + \Phi_{yy}\Phi_y^2] = 0, \tag{32.62}$$

which we may write in the form

$$Q' + AQ + B = 0 \quad \text{on} \quad \Psi = 0 \tag{32.63}$$

where A and B are independent of t. Division by $\Phi_x^2 + \Phi_y^2$ is possible since (32.59) implies that this does not vanish. Note also that

$$B = g\frac{\Phi_y}{\Phi_x^2 + \Phi_y^2} = g\, y_{\Phi}\big|_{\Psi=0} = g\frac{d}{d\Phi}\eta, \tag{32.64}$$

and that both A and B may be considered as functions of Φ. Consider two cases: (a) $A =$ const., (b) $A \neq$ const. (a) In this case, since B is independent of t and $Q' + AQ$ is independent of Φ, it follows from (32.63) that both equal constants. It now follows from (32.64) that, unless this constant is zero, the profile $\eta(x)$ will be unbounded and the first part of (32.59) will be contradicted. Hence, in case (a) $\eta =$ const. and the mapping F must be of the form $F = az + b$, a and b real. It then follows that $A = 0$ and hence $Q' = 0$, i.e. the flow is uniform. (b) Let A_1, A_2 be two different values of A, $A_1 \neq A_2$. Write equation (32.63) for each value and subtract. This yields

$$Q = -\frac{B_1 - B_2}{A_1 - A_2}. \tag{32.65}$$

But then Q is evidently independent of t. Hence also $f(\zeta,t) = QF(\zeta)$ is also independent of t. This completes the proof.

The next theorem, due to Dubreil-Jacotin [1932], specifically requires that the fluid be heterogeneous.

__Theorem__: Suppose the motion of an incompressible heterogeneous fluid to be irrotational, the free surface to move without change of form, and that, in a coordinate system moving with the surface, conditions (32.59) are satisfied. Then not only is the velocity field steady, but also E, p and ρ are constant along the streamlines.

It follows from the preceding theorem that the velocity is steady, hence that $E = E(x,y)$. However, we may still conceivably have $p = p(x,y,t)$, $\rho = \rho(x,y,t)$. The equations (32.54) may now be written in the form

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial E}{\partial x}, \quad -\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial E}{\partial y}. \tag{32.66}$$

Elimination of first ρ, then p between these two equations yields

$$\frac{\partial(p,E)}{\partial(x,y)} = 0, \qquad \frac{\partial(\rho,E)}{\partial(x,y)} = 0$$

We assume that the corresponding functional relations may be solved and write

$$p = p(E,t), \qquad \rho = \rho(E,t), \tag{32.67}$$

where, from (32.66)

$$\rho = -\frac{\partial p}{\partial E}. \tag{32.68}$$

From the equation expressing incompressibility, namely,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0,$$

follows

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial E}\left(u\frac{\partial E}{\partial x} + v\frac{\partial E}{\partial y}\right) = \frac{\partial \rho}{\partial t} + \frac{\partial(E,\psi)}{\partial(x,y)}\frac{\partial \rho}{\partial E} = 0 \qquad (32.69)$$

We shall assume $\partial\rho/\partial E \neq 0$ everywhere, and may thus write

$$\frac{\partial(E,\psi)}{\partial(x,y)} = -\frac{\rho_t(E,t)}{\rho_E(E,t)}. \qquad (32.70)$$

Since the left-hand side is independent of t, it follows from the form of the right-hand side that we may set both sides equal to $k(E)$, i.e.

$$\frac{\partial \rho}{\partial t} + k(E)\frac{\partial \rho}{\partial E} = 0. \qquad (32.71)$$

Let us suppose $k(E) \not\equiv 0$, e.g. $k(E_1) \neq 0$. Then ρ must be a function of the form

$$\rho = \rho\left(t - \int_{E_1}^{E}\frac{dE}{k(E)}\right) \qquad (32.72)$$

in some neighborhood of E_1. If $k(E)$ vanishes for some values of E, let E_0 be the first zero larger than E_1. Then from (32.71) and (32.72)

$$\rho_t(E,t) = \rho'\left(t - \int_{E_1}^{E}\frac{dE}{k(E)}\right) \to 0 \quad \text{as } E \to E_0. \qquad (32.73)$$

But (32.73) can hold for all t only if $\rho' = 0$, i.e. if $\rho =$ const., which is contrary to the assumed heterogeneity. Moreover, at least one such zero of k exists, for we already

know from (32.60) that E is constant along the free surface, so that in steady motion the Jacobian in (32.70) vanishes for $\psi = 0$. Hence $A(E) = 0$ for the corresponding value of E. We must conclude that $A(E) \equiv 0$. This implies, from (32.70) that $E = E(\psi)$ and $\rho = \rho(E)$. From (32.68) and the condition $p_t = 0$ on the free surface, it follows that also $p = p(E)$. Hence p, ρ and E are all constant on streamlines.

The last in this complex of theorems is also due to Dubreil-Jacotin [1932].

<u>Theorem</u>: There cannot exist irrotational waves in a heterogeneous fluid such that the profile is propagated without change of form.

This follows immediately from the first and last theorems proved above, and is, of course, subject to condition (32.59). This striking result is all the more s-- in view of the fact that Gerstner's wave (section 34β) does ...de a steadily propagating wave, even in a heterogeneous fluid. The theorem also casts some doubt upon the significance of the linearized theory of irrotational wave motion in a heterogeneous fluid as developed, for example, in Lamb [1932, § 235]. Such a wave evidently cannot be considered as a first approximation to an exact steady solution.

32γ. <u>Some transformations of the boundary-value problem</u>

By means of introduction of new variables or other devices, it is possible to formulate the boundary-value problem for exact solutions in a variety of ways. Several such formulations will be considered in section 34α on inverse methods. Here we give a few which seem to be of general interest.

Inversion of $f(z)$. One elementary but important transformation has already been introduced in section 32α in the discussion of mass transport. This is the inversion of the velocity potential $f(z)$ when $|f'|$ vanishes nowhere within the fluid, and treatment of f as the independent variable. This has the advantage that under certain circumstances the domain of definition of the independent variable can be given exactly; when z is the independent variable, the domain of definition is one of the unknowns of the problem. For example, if the motion is reducible to a steady flow with discharge rate Q, one may take the surface profile to correspond to $\psi = 0$ and the bottom streamline to correspond to $\psi = -Q$. Hence the domain of definition of $z(f)$ is the strip $0 \geq \psi \geq -Q$; if the fluid is infinitely deep, the domain is the half-plane $\psi \leq 0$. Whenever f can be taken as the independent variable, then one can also express $w = f'$ as a function of f. It has been established independently by Gerber [1951] and Lewy [1952a] that the equation describing the free surface, $z = z(\varphi)$, is an analytic function of φ at all points for which $w \neq 0$.

Stokes' "second method". In the introduction to section 27 it was mentioned that Stokes [1880], in a supplement to an earlier paper in his collected works, developed a method for approximating exact periodic waves which is different from the straightforward generalization of infinitesimal-wave theory expounded in that section. This method is based upon use of f as the independent variable and expansion of z as a Fourier series in f:

$$c\mathfrak{z} = f + i\frac{c\lambda}{2\pi}\sum_{n=0}^{\infty} a_n e^{-in2\pi f/c\lambda} \tag{32.74}$$

or

$$c\mathfrak{z} = f + i\frac{c\lambda}{2\pi}a_0 + \frac{c\lambda}{2\pi}\sum_{n=1}^{\infty} a_n \sin n\frac{2\pi}{c\lambda}(f + iG) \tag{32.75}$$

for infinite and finite depth respectively; the a_n may be taken to be real. Here ψ is taken as in the preceding paragraph. The coefficients a_n are to be determined from the condition that the pressure be constant on the surface, i.e. from

$$q^2 + 2gy = C \quad \text{for} \quad \psi = 0. \tag{32.76}$$

If the mean water level is taken at $y = 0$ and the fluid is infinitely deep, then $C = c^2$; we shall consider only this case here. Then equation (32.76) may be expressed as

$$(c^2 - 2gy)|\mathfrak{z}'|^2 = 1. \tag{32.77}$$

Substitution of (32.74) in (32.77) yields

$$\left(1 - \frac{g\lambda}{\pi c^2}\sum_{n=0}^{\infty} a_n \cos\frac{2\pi n\phi}{c\lambda}\right)\left(1 + 2\sum_{n=1}^{\infty} n a_n \cos\frac{2\pi n\phi}{c\lambda} + \sum_{n,m=1}^{\infty} nm\, a_n a_m \cos(n-m)\frac{2\pi\phi}{c\lambda}\right) = 1.$$

(32.78)

After multiplying the two factors and reducing the cosine products to cosines of sums and differences, the resulting expression may be put into the form

$$\sum_{r=0}^{\infty}\left(\frac{g\lambda}{\pi c^2} b_n + c_n\right)\cos\frac{2\pi r\phi}{c\lambda} = 0. \qquad (32.79)$$

where the b_n's and c_n's are forms of the third degree in the a_n's. The coefficients of the individual cosine terms must then be equated to zero. This results in an infinite sequence of equations, each involving all the a_n's and $g\lambda/\pi c^2$. In order to proceed further, one must devise some method for approximate determination of the a_n's. Stokes' procedure was to assume that each a_n could be expanded in a power series in some parameter, the initial term in the series having the power r. This allows one to carry through a step-by-step improvement in the approximation of the a_n's by including successively higher powers of the parameter. We shall not pursue the matter further, but remark that the most systematic arrangement of such computations seems to have been devised by Sretenskii [1952].

Levi-Civita's differential-difference equation. The following theorem, due to Levi-Civita [1907], reduces determination of $w(f)$ for steady flow over a horizontal bottom to solution of a differential-difference equation.

Theorem: The complex velocity $w = u - iv$ of an irrotational gravity flow with constant discharge rate Q and with $u \geq \varepsilon > 0$ must satisfy the differential-difference equation

$$\frac{d}{df}\left[w(f+iQ)\,w(f-iQ)\right] - ig\left[\frac{1}{w(f+iQ)} - \frac{1}{w(f-iQ)}\right] = 0. \qquad (32.80)$$

Conversely, any function $w(f)$ satisfying (32.80) which is regular in the strip $-2Q \le \mathrm{Im}\, f \le 0$, finite at ∞, real on $\mathrm{Im}\, f = -Q$ and has $u > c > 0$ represents such a flow.

In order to derive (32.80), we note first that the functions $w(f)$ and $z(f) + if$ both have vanishing real parts for $\psi = -Q$ and consequently can be extended by reflection to the strip $-2Q \le \psi \le -Q$

$$w(\bar{f} - 2iQ) = \overline{w(f)} \qquad z(\bar{f} - 2iQ) - if = \overline{z(f) - if} \tag{32.81}$$

The free-surface condition may be expressed by the equation

$$\frac{\partial}{\partial \phi} w\bar{w} + 2g \frac{\partial}{\partial \phi} y = 0 \quad \text{for} \quad \psi = 0 \tag{32.82}$$

or, by making use of the extended definitions of w and z, by

$$\frac{\partial}{\partial \phi}\left\{ w(\phi) w(\phi - 2iQ) - ig\left[z(\phi) - z(\phi - 2iQ)\right]\right\} = 0 \tag{32.83}$$

Consider the function

$$H(f) = w(f + iQ)\, w(f - iQ) - ig\left[z(f + iQ) - z(f - iQ)\right]. \tag{32.84}$$

Evidently, H is defined and is regular on the line $\psi = -Q$ and thus in some neighborhood of this line. From (32.83) it follows that $H'(\phi - iQ) = 0$, hence that $H'(f) \equiv 0$ in its region of definition. Equation (32.80) follows from the fact that $z'(f \pm iQ) = 1/w(f \pm iQ)$. For proof of the converse we refer to Levi-Civita's paper. Levi-Civita also gives a special form of (32.80) appropriate to a space-periodic flow. Cisotti [1919] generalized the preceding theorem to include a variable discharge rate. The equation (32.80) may be considered to contain the equation (22.30),

when in that equation $f(z,t) = f(z - ct)$, in the sense that linearization of (32.80) by assuming

$$w = c(1 + \varepsilon w_1 + \cdots)$$

yields (22.30).

Rudzki's transformation. The following transformation was apparently first introduced by Rudzki [1898]. It has later been used by many others in the investigation of exact water waves. The validity of the reformulated boundary condition is not limited to periodic waves. However, it is assumed that a coordinate system has been selected with respect to which the flow is steady. It is again assumed that $u > \varepsilon > 0$. Let θ be the angle between the velocity vector $\overline{w} = u + iv$ and the positive x-axis. Then one may write

$$w = u - iv = q e^{-i\theta} = c e^{-i\omega} \tag{32.85}$$

where

$$\omega = \theta + i\tau, \qquad q = c e^{\tau}. \tag{32.86}$$

Here c is some typical velocity, say the wave velocity as defined in section 7. We consider ω as a function of f and let $\psi = 0$ correspond to the free surface. The free-surface condition may then be expressed by

$$g \frac{\partial y}{\partial \phi} + q \frac{\partial q}{\partial \phi} = 0 \quad \text{for } \psi = 0. \tag{32.87}$$

But (see 32.16)

$$\frac{\partial y}{\partial \phi} = \frac{1}{q^2} \frac{\partial \psi}{\partial y} = \frac{1}{q} \sin\theta$$

and, from (32 86),

$$\frac{\partial q}{\partial \phi} = c e^{\tau} \frac{\partial \tau}{\partial \phi} = q \frac{\partial \tau}{\partial \phi} .$$

Hence (32.87) becomes

$$\frac{\partial \tau}{\partial \phi} = - \frac{g}{q} \frac{1}{q^3} \sin \theta = - \frac{g}{c^3} e^{-3\tau} \sin \theta \quad \text{for } \psi = 0, \tag{32.88}$$

or, since $\partial\tau/\partial\phi = -\partial\theta/\partial\psi$ from the Cauchy-Riemann equations,

$$\frac{\partial \theta}{\partial \psi} = \frac{g}{c^3} e^{-3\tau} \sin \theta \quad \text{for } \psi = 0. \tag{32.89}$$

It one can find a function $\omega(f)$ regular in the strip $0 \geq \psi \geq -Q$, with $|\theta| < \frac{1}{2}\pi - \varepsilon'$, and with its real and imaginary parts satisfying (32.88) or (32.89) on $\psi = 0$, one may then construct from it a free-surface flow with gravity. Of course, further conditions must be imposed at $\psi = -Q$ or as $\psi \to -\infty$.

<u>Nekrasov's transformation</u>. The following transformation is due to Nekrasov [1921, 1951]. It will be assumed that the surface is periodic with period λ, symmetric about a crest and that the fluid is infinitely deep and $\lim_{y \to -\infty} w = c$. Let the origin in the z-plane be taken at a crest, $\psi = 0$ be the free surface, and assume $u > \varepsilon > 0$. In addition to the z- and f-planes, we introduce a ζ-plane,

$$\zeta = \xi + i\eta = \rho e^{i\gamma}, \tag{32.90}$$

related to the f-plane through

$$\zeta = \exp \frac{-2\pi i}{\lambda c} f . \tag{32.91}$$

With a cut along the negative ξ-axis there is a one-to-one correspondence between the various domains CAOBD shown in Figure 46.

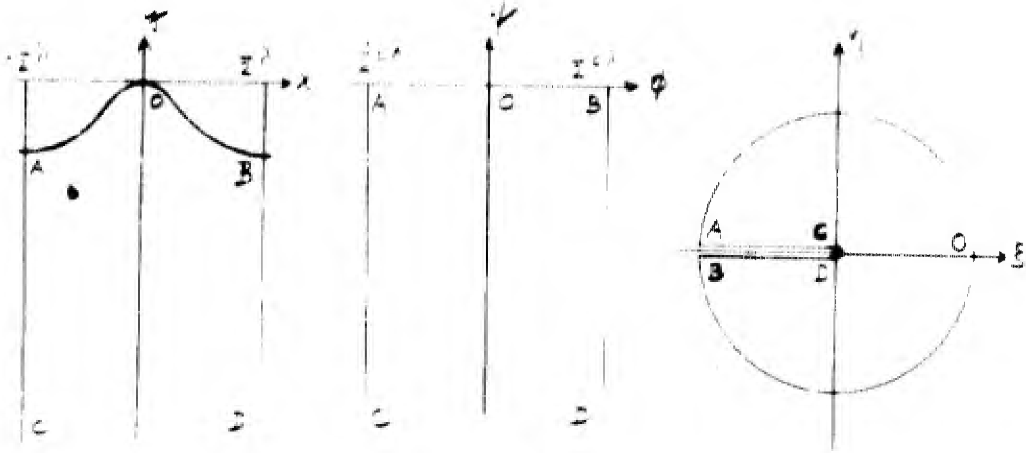


FIGURE 46

The relation between the z- and ζ-planes will be determined by

$$\frac{dz}{d\zeta} = -\frac{\lambda}{2\pi i}\,\frac{h(\zeta)}{\zeta}, \quad h(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots, \quad a_k \text{ real} \tag{32.92}$$

where $h(\zeta)$ is regular in the disc and is related to ω by

$$\omega = \frac{df}{d\zeta}\,\frac{d\zeta}{dz} = \frac{c}{h(\zeta)}. \tag{32.93}$$

The form of h shown in (32.92) follows from the assumed properties of the motion. Since $\rho = 1$ on the free surface, the condition of constant pressure may be expressed by

$$2g\frac{\partial y}{\partial \gamma} + \frac{\partial q^2}{\partial \gamma} = 0 \quad \text{for } \rho = 1. \tag{32.94}$$

But

$$\left.\frac{\partial y}{\partial \gamma}\right|_{\rho=1} = \operatorname{Im} \left.\frac{dz}{d\zeta}\frac{d\zeta}{d\gamma}\right|_{\rho=1} = \operatorname{Im} \left.\frac{-\lambda}{2\pi i}\frac{h(\zeta)}{\zeta} i\zeta\right|_{\rho=1}$$

$$= \frac{-\lambda}{2\pi} \operatorname{Im} h(e^{i\gamma}). \tag{32.95}$$

It then follows from (32.93) and (32.95) that

$$\frac{d}{d\gamma}\frac{1}{h(e^{i\gamma})h(e^{-i\gamma})} = \frac{\lambda g}{\pi c^2} \operatorname{Im} h(e^{i\gamma}). \tag{32.96}$$

In this formulation of the problem one seeks a function $h(\zeta)$, regular in the disc $|\zeta| \le 1$, real on the real axis, $h(0)=1$, and satisfying (32.96). From such a function one can easily construct a periodic gravity flow with free surface.

Nekrasov's integral equation. Nekrasov also considers the function ω of (32.92), but as a function of ζ. Let us start from (32.88) and compute

$$\frac{\partial \tau}{\partial \gamma} = \frac{\partial \tau}{\partial \phi}\frac{\partial \phi}{\partial \gamma} = -\frac{g}{c^3} e^{-3\tau} \sin\theta \cdot \frac{-\lambda c}{2\pi} = \frac{g\lambda}{2\pi c^2} e^{-3\tau} \sin\theta \quad \text{for } \rho = 1. \tag{32.97}$$

One may formally integrate this equation and obtain

$$e^{3\tau} = \frac{3}{2\pi}\frac{g\lambda}{c^2\mu}\left[1 + \mu\int_0^{\gamma} \sin\theta(\alpha)\, d\alpha\right], \tag{32.98}$$

where $1/\mu$ is the integration constant; μ is related to the velocity at the crest, $q_0 = \tau(1) = c/h(1)$, by

$$\mu = \frac{3}{2\pi}\frac{g\lambda c}{q_0^3} > 0. \tag{32.99}$$

Substitution of (32.98) into (32.99) yields the following equation for the relation between τ and θ on the boundary:

$$\frac{d\tau(\gamma)}{d\gamma} = \frac{1}{3}\,\frac{\mu \sin\theta(\gamma)}{1+\mu\int_0^{\gamma}\sin\theta(\alpha)\,d\alpha} \tag{32.100}$$

(it follows from (32.98) that the denominator does not vanish). It is known from the theory of functions of a complex variable [see, e.g., Carathéodory, Funktionentheorie, Bd. 1, § 147-9, Birkhäuser, Basel, 1950] that, if a function is regular within and on a closed Jordan curve, it is determined up to an additive constant by giving either its real or imaginary part on the boundary. In particular, in the case at hand we may express the value of θ on the boundary $|\zeta| = 1$ in terms of τ on the boundary:

$$\theta(\gamma) = \text{const} - \frac{1}{2\pi}\,\mathrm{Pr}\int_0^{2\pi}\tau(\beta)\cot\tfrac{1}{2}(\gamma-\beta)\,d\beta \tag{32.101}$$

where the constant $= \theta|_{\zeta=0} = 0$. An integration by parts gives

$$\theta(\gamma) = -\frac{1}{\pi}\int_0^{2\pi}\frac{d\tau}{d\beta}\log\left|\sin\tfrac{1}{2}(\gamma-\beta)\right|d\beta. \tag{32.102}$$

From the assumed symmetry about a crest follows $\tau'(-\beta) = -\tau'(\beta)$, so that (32.102) may be expressed as follows:

$$\theta(\gamma) = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\tau}{d\beta}\log\left|\frac{\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\beta)}\right|d\beta. \tag{32.103}$$

Substitition of (32.100) into (32.103) yields Nekrasov's nonlinear integral equation for $\theta(\gamma)$:

$$\theta(\gamma) = \frac{1}{6\pi}\int_0^{2\pi}\frac{\mu\sin\theta(\beta)}{1+\mu\int_0^{\beta}\sin\theta(\alpha)\,d\alpha}\log\left|\frac{\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\beta)}\right|d\beta. \tag{32.104}$$

If one can find θ satisfying (32.104), one can then reconstruct $\omega(\zeta)$ and hence the whole flow.

Nekrasov [1928, 1951] carried through a similar analysis when the depth is finite. We shall only sketch it. In Figure 35 suppose that $y=-h_1$ represents the bottom (h_1 is not the mean depth) and $\psi=-Q$ the corresponding streamline. In the ζ-plane this maps into a circle of radius $\rho_0 < 1$, where

$$\rho_0 = e^{-\frac{2\pi Q}{\lambda c}} . \tag{32.105}$$

In (32.92) $h(\zeta)$ becomes a Laurent series. The integral equation for $\theta(\gamma)$ remains the same in form as (32.104), but the kernel function $\log|\cdots|$ is now replaced by

$$\sum_{n=1}^{\infty} \frac{2}{n} \tanh \frac{2\pi Q}{\lambda c} \sin n\gamma \sin n\beta \tag{32.106}$$

Moiseev [1957b] has further generalized Nekrasov's equation so as to allow a wavy bottom.

The solution $\theta(\gamma)$ of (32.104) will, of course, depend upon the parameter μ, except for the trivial solution $\theta \equiv 0$ corresponding to a uniform flow. It is possible to show that not all μ's are allowable. Let

$$M = \max|\theta(\gamma)| . \tag{32.107}$$

It then follows from (32.102) that

$$\begin{aligned} 0 \le |\theta(\gamma)| &< \frac{1}{6\pi} \frac{\mu \sin M}{1-\pi\mu\sin M} \int_0^{2\pi} -\log\left|\sin\tfrac{1}{2}(\gamma-\beta)\right| d\beta \\ &< \frac{1}{3} \frac{\mu \sin M}{1-\pi\mu \sin M} , \end{aligned} \tag{32.108}$$

hence that

$$0 \leq M < \frac{1}{3}\,\frac{\mu \sin M}{1 - \pi\mu \sin M}\,. \tag{32.109}$$

From this follows

$$\frac{1}{\pi \sin M} > \mu > \frac{1}{\pi \sin M + \frac{1}{3} M^{-1} \sin M} > \frac{3}{1+3\pi}\,. \tag{32.110}$$

Villat's integral equation. Even though we shall not consider its contents in any detail, it would be improper not to mention an important paper of Villat [1915]. Villat wished to find the steady motion of a fluid in a canal of given bottom profile and also with a given top profile over the part of the fluid upstream of some point. Downstream of this point the top profile is one of constant pressure. The boundary condition on the free surface, (32.89), is modified by introduction of new variables, and a pair of integral equations, one of them nonlinear, is derived. The method is also applicable if the upstream "cover" is absent and, in fact, becomes a little simpler. The chief use made of the procedure by Villat is as an inverse method in which the free surface is given and the corresponding bottom and cover determined.

33. Waves of maximum amplitude

In the higher-order theory of infinitesimal waves one of the important effects of including higher-order terms was to make the profile more peaked at the crests and flatter in the troughs. The effect was the same for either steady progressive waves or standing waves. Since the peakedness increased with increase of

the amplitude-to-wave-length ratio, it seems reasonable to conjecture that there is some bound to this ratio and that, if a wave of maximum amplitude-to-length ratio exists, it will be characterized by a corner or a cusp at the crest, at least if capillarity is neglected. It has never been proved that such waves exist. However, if one assumes their existence, it is possible to prove some necessary properties. This will be done below.

Following an earlier erroneous investigation of Rankine [1865], Stokes [1880, p. 225] showed that, if a corner occurs in steady motion, the angle included between the tangents must be 120°. Michell [1893] assumed that a periodic highest progressive wave exists and showed how to compute the coefficients of an associated series, but without proving convergence. Havelock [1919] made Michell's procedure the basis of a general method of approximation to periodic progressive waves, again with no proof of convergence. Michell's wave was later investigated by a different procedure by Nekrasov [1920]. However, Nekrasov did not carry his computations to the same degree of accuracy as Michell and Havelock, so that the numerical results are discrepant. More recently Yamada [1957] rediscovered Nekrasov's method and carried through the calculations with the necessary accuracy; the results are now in substantial agreement with those of Havelock and Michell.

Penney and Price [1952b], in their work on standing waves of finite amplitude, include an analysis intended to show that, if there exists a standing wave of maximum amplitude with a corner at the crest, then the angle must be 90°. G.I. Taylor[1953]

has questioned the validity of the proof, and it appears, in fact, to be untenable. On the other hand, in the same paper Taylor reports the results of experiments which appear to confirm Penny and Price's prediction. In view of the present unsatisfactory state of the theory, it will not be further discussed here.

Stokes' theorem. We prove first Stokes' theorem on the angle at a corner in steady flow. Let the corner be at the origin $z=0$, the free surface be the streamline $\psi=0$, and $\phi=0$ at the corner. Since $z=0$ is assumed to be a corner, it must also be a stagnation point and the constant-pressure condition on the surface may be taken in the form

$$q^2 + 2g\eta(x) = 0. \tag{33.1}$$

In the mapping from the z- to the f-plane the point $z=0$ must be a branch point, so that in the neighborhood of $z=0$ the complex velocity potential will take the form

$$w = Az^n. \tag{33.2}$$

If $\alpha_+ < 0$ is the angle between the right-hand tangent to the corner and OX, then near $z=0$ equation (33.1) can be written

$$|A|^2 n^2 r^{2n-2} + 2gr\sin\alpha_+ = 0.$$

This can hold for all small r only if

$$n = 2/3, \quad |A| = \tfrac{3}{2}\sqrt{g}. \tag{33.3}$$

It also follows that, if α_- is the angle between the left-hand tangent and OX, then $\sin\alpha_- = \sin\alpha_+$ and $\alpha_- = 180^\circ - \alpha_+$ so that the surface is symmetrical about OX near the corner. If $\psi \leq 0$ corresponds to the region occupied by fluid and if the branch of

$z = re^{i\alpha}$ with $-\frac{3}{2}\pi < \alpha < \frac{1}{2}\pi$ is taken, then the complex velocity potential has the following form near $z = 0$:

$$f(z) = -\frac{2}{3}\sqrt{\tfrac{g}{?}}\,(-iz)^{3/2}$$

$$= -\frac{2}{3}\sqrt{\tfrac{g}{?}}\, r^{3/2}\left[\cos\tfrac{3}{2}(\alpha - \tfrac{1}{2}\pi) + i\sin\tfrac{3}{2}(\alpha - \tfrac{1}{2}\pi)\right]. \qquad (33.4)$$

The streamline $\psi = 0$ has a corner at $z = 0$ with included angle 120°. In this case the flow is to the right. The inversion of (33.4) gives

$$z(f) = \left[\frac{3}{2\sqrt{g}}\right]^{2/3} e^{-i\pi/6} f^{2/3} \qquad (33.5)$$

for f near 0.

33a . Periodic wave of maximum height

Let us suppose that a periodic progressive wave of maximum amplitude-length ratio exists. We may take this as a steady flow with complex velocity potential $f(z) = \phi + i\psi$ and with

$$\lim_{z \to -\infty} f'(z) = c. \qquad (33.6)$$

Let the origin of the z-plane be at one of the crests, the surface profile correspond to $\psi = 0$, and the origin of the f-plane to that of the z-plane. Then the free surface condition may be taken in the form (33.1).

Michell's method. First we give Michell's procedure for finding $f'(z)$. As we have done earlier, we shall write

$$f'(z) = qe^{-i\theta}, \qquad z'(f) = \frac{1}{q}e^{i\theta}. \qquad (33.7)$$

From the assumed periodicity and symmetry, θ is an odd periodic function of ϕ with period $c\lambda$ for $\psi = 0$. From (33.7) follows

$$\frac{d}{df}\log \zeta'(f) = -\frac{\partial}{\partial\phi}\log \xi + i\frac{\partial\theta}{\partial\phi} \tag{33.8}$$

For $\psi = 0$, $\partial\theta/\partial\phi$ is an even periodic function of ϕ with removable singularities at the crests: we expand it in a Fourier series:

$$\frac{\partial\theta}{\partial\phi} = \frac{\pi}{c\lambda}\left[a_0 + a_1\cos\frac{2\pi\phi}{c\lambda} + a_2\cos\frac{4\pi\phi}{c\lambda} + \cdots\right]. \tag{33.9}$$

The a_k are real. Substitute (33.9) into (33.8) and rewrite it in the following way:

$$\left[\frac{d}{df}\log \zeta'(f) - i\frac{\pi}{c\lambda}\sum_{n=0}^{\infty} a_n e^{-i2n\pi f/c\lambda}\right]_{\psi=0}$$

$$= -\frac{\partial}{\partial\phi}\log \xi - \frac{\pi}{c\lambda}\sum_{n=1}^{\infty} a_n \sin\frac{2n\pi\phi}{c\lambda}. \tag{33.10}$$

Now consider the function

$$Z(f) = \frac{d}{df}\log \zeta'(f) - i\frac{\pi}{c\lambda}\sum_{n=0}^{\infty} a_n e^{-i2n\pi f/c\lambda} \tag{33.11}$$

Z(f) is defined in the whole lower half-plane, is regular for $\psi < 0$, and, as $\psi \to -\infty$, $Z(f) \to -i\pi a_0/c\lambda$. Moreover, from (33.10) Z is also real on the real axis and hence may be extended by reflection to the upper half-plane. Z is then a function with only singularities on the real axis at the points $\phi = nc\lambda$ associated with the crests. The form of the singularity may be determined from (33.5). In fact, near $f = 0$

$$\frac{d}{df}\log \zeta' = -\frac{1}{3}\frac{1}{f}. \tag{33.12}$$

Hence Z(f) has singularities of the form

$$-\frac{1}{3}\,\frac{1}{f-nc\lambda}\,, \quad n=0,\pm1,\pm2,\ldots, \tag{33.13}$$

along the real axis, and only these, so that it must have the form

$$Z(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda} + b\,, \quad b = \text{const.} \tag{33.14}$$

From (33.14) $Z \to b - i\pi/3c\lambda$ as $\psi \to -\infty$. Then from (33.11)

$$b = -i\frac{\pi}{c\lambda}\left(a_0 - \frac{1}{3}\right).$$

Since Z must be real for real f, it follows that

$$a_0 = \frac{1}{3} \tag{33.15}$$

and

$$Z(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda}. \tag{33.16}$$

It now follows from the definition of Z(f) that

$$\frac{d}{df}\log z'(f) = -\frac{\pi}{3c\lambda}\cot\frac{\pi f}{c\lambda} + i\frac{\pi}{3c\lambda} + i\frac{\pi}{c\lambda}\sum_{n=1}^{\infty} a_n e^{-i2n\pi f/c\lambda}, \tag{33.17}$$

which yields, after integration, inversion of the logarithm and use of (33.6) to evaluate a multiplicative constant,

$$\begin{aligned} z'(f) &= \frac{1}{c\sqrt[3]{2}}\, e^{\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{-\frac{1}{3}} \prod_{n=1}^{\infty}\exp\left(\frac{-i\lambda}{2n\pi}a_n e^{-i2n\pi f/c\lambda}\right) \\ &= \frac{1}{c\sqrt[3]{2}}\, e^{\frac{1}{3}i\pi f/c\lambda}\left(i\sin\frac{\pi f}{c\lambda}\right)^{-\frac{1}{3}} \sum_{n=0}^{\infty} b_n e^{-i2n\pi f/c\lambda}, \end{aligned} \tag{33.18}$$

where $b_0 = 1$ and the b_n are real. The branch of the root must be chosen so that its argument lies between $\pm\frac{1}{2}\pi$ for $\psi = 0$. From (33.18) one finds immediately also

$$f'(z) = c\sqrt[3]{2}\, e^{-\frac{1}{3}i\pi f/c\lambda}\left(i \sin \frac{\pi f}{c\lambda}\right)^{\frac{1}{3}} \sum_{n=0}^{\infty} c_n e^{-i2n\pi f/c\lambda}, \quad c_0 = 1,\ c_n \text{ real} \tag{33.19}$$

Aside from the first one, the coefficients in (33.18) or (33.19) are still to be determined. The constant-pressure condition for the surface profile is still available for this purpose, for we have made use of the equation (33.4) or (33.5) only through the value of the exponent. The value of the gravitation constant has not entered into (33.18) or (33.19). In fact, a comparison of (33.5) after differentiation and (33.18) in the neighborhood of $f = 0$ yields immediately

$$\frac{c^2}{g\lambda} = \frac{3}{4\pi}\left[1 + b_1 + b_2 + \cdots\right]^3 = \frac{3}{4\pi}\left[1 + c_1 + c_2 + \cdots\right]^{-3}, \tag{33.20}$$

so that, once the b_n or c_n are determined, the relation between wave length and velocity may be found. This method could presumably be pursued to obtain a sequence of further equations for determination of the b_n. However, Michell proceeds somewhat differently. If we differentiate (33.1) with respect to ϕ, we may write the free surface condition as follows (cf. (32.54) and following):

$$\frac{\partial}{\partial \phi} q^2 = -\frac{g}{q^2}\frac{\partial \phi}{\partial y} \quad \text{for } \psi = 0$$

or (33.21)

$$\frac{\partial}{\partial \phi}|f'|^4 = 4g \,\mathrm{Im}\, f' \quad \text{for } \psi = 0.$$

Substitution of (33.19) in (33.21) yields an equation of the following form

$$\frac{4}{3}\pi\, 2^{4/3}\frac{c^3}{\lambda}\sin^{1/3}\frac{\pi\phi}{c\lambda}\left\{A_1\cos\frac{\pi\phi}{c\lambda}+A_3\cos\frac{3\pi\phi}{c\lambda}+\cdots\right\}=$$

(33.22)

$$\frac{4}{3}gc\sin^{1/3}\frac{\pi\phi}{c\lambda}\left\{B_1\cos\frac{\pi\phi}{c\lambda}+B_3\cos\frac{3\pi\phi}{c\lambda}+\cdots\right\},$$

where the B_n's depend linearly upon the c_n's , and the A_n's depend upon them in a more complicated manner. The derivation of (33.22), especially of the right-hand part, and of the particular dependence of the A_n's and B_n's upon the c_n's is rather tedious and we refer to either Michell's original paper or preferably to Havelock's more general and systematic treatment. Equating coefficients of the individual cosine terms leads to a set of equations relating $c^2/g\lambda$, c_1 , c_2 ,.... The values as computed by Havelock, which we assume to be somewhat more accurate than Michell's own, are as follows:

$$\frac{g\lambda}{c^2}=0.833\cdot 2\pi\,,\; c_1=0.0414,\; c_2=0.0114,\; c_3\;0.0042,\; c_4=0.0014. \tag{33.23}$$

The value for $g\lambda/c^2$ should be compared with that for infinitesimal waves, namely 2π. Substitution of $\frac{1}{2}c\lambda$ for ϕ in (33.19) yields the velocity at a trough:

$$u=c\sqrt[3]{2}\,[1-c_1+c_2-c_3+\cdots]\doteq 1.219c. \tag{33.24}$$

From $q^2+2g\eta=0$ one may now find η for the trough and hence the amplitude-wave length ratio:

$$\left|\frac{\eta}{\lambda}\right|=\frac{1}{\sqrt[3]{2}}\frac{c^2}{g\lambda}[1-c_1+c_2-\cdots]^2\doteq 0.1418. \tag{33.25}$$

H. Jeffreys [1951] has recently reexamined the basis of the Michell-Havelock method of approximation and concludes that an apparent discrepancy between the values in (33.23) and equation (33.20) does not really indicate a numerical error in the computations.

We note in passing that Michell also gave the form of $f'(\mathfrak{z})$ analogous to (33.19) which must hold if a highest wave with corner exists in a fluid of finite depth.

Method of Nekrasov and Yamada. This method makes use of the ζ-plane introduced in (32.57) and related to the f-plane by (32.58). We may again make use of Figure 46 but must keep in mind that in the z-plane there is now a corner at O with an included angle of 120°. Hence (32.59), the equation relating the $\mathfrak{z}$- and ζ-planes, must be replaced by

$$\frac{d\mathfrak{z}}{d\zeta} = -\frac{\lambda}{2\pi i}\,\frac{h(\zeta)}{\zeta(1-\zeta)^{1/3}}\,,\quad h(\zeta) = 1 + a_1\zeta + a_2\zeta^2 + \cdots \tag{33.26}$$

and (32.60) by

$$w = \frac{df}{d\mathfrak{z}} = c\,\frac{(1-\zeta)^{1/3}}{h(\zeta)}\,. \tag{33.27}$$

The coefficients a_n are now to be determined by the constant-pressure condition on the free surface taken in the form (32.61). From

$$q^2\big|_{\rho=1} = c^2\,\frac{\left[(1-e^{i\gamma})(1-e^{-i\gamma})\right]^{1/3}}{h(e^{i\gamma})\,h(e^{-i\gamma})} = c^2\,\frac{\left(2\sin\frac{1}{2}\gamma\right)^{2/3}}{h(e^{i\gamma})\,h(e^{-i\gamma})} \tag{33.28}$$

and

$$\left.\frac{dz}{d\gamma}\right|_{\rho=1} = -\frac{\lambda}{2\pi}\frac{h(e^{i\gamma})}{(1-e^{i\gamma})^{1/3}}$$

$$= -\frac{\lambda}{2\pi}\left(2\sin\tfrac{1}{2}\gamma\right)^{-\frac{1}{3}} e^{-i(\gamma-\pi)/6}\, h(e^{i\gamma}) \qquad (33.29)$$

one obtains as the equation analogous to (32.63)

$$\frac{d}{d\gamma}\,\frac{\left(2\sin\frac{1}{2}\gamma\right)^{2/3}}{h(e^{i\gamma})\,h(e^{-i\gamma})} = \frac{g\lambda}{\pi c^2}\left(2\sin\tfrac{1}{2}\gamma\right)^{-1/3} \operatorname{Im}\left\{e^{-i(\gamma-\pi)/6}\, h(e^{i\gamma})\right\}.$$

(33.30)

This yields a set of equations for determination of $g\lambda/c^2$, a_1, a_2, $\cdots$. The actual computation appears to be as tedious as that of Michell's method and, in fact, Nekrasov's [1920] computations do not seem to have yielded as accurate results as Michell's. However, as mentioned earlier, Yamada [1957] has set up a systematic computation procedure and has obtained results in substantial agreement with those of Michell and Havelock. Once $g\lambda/c^2$ and the a_n have been determined, the surface profile can be found in parametric form by integrating (33.29) with respect to γ from 0 to γ. Figure 47, reproduced from Yamada's cited paper, shows the form of the profile.

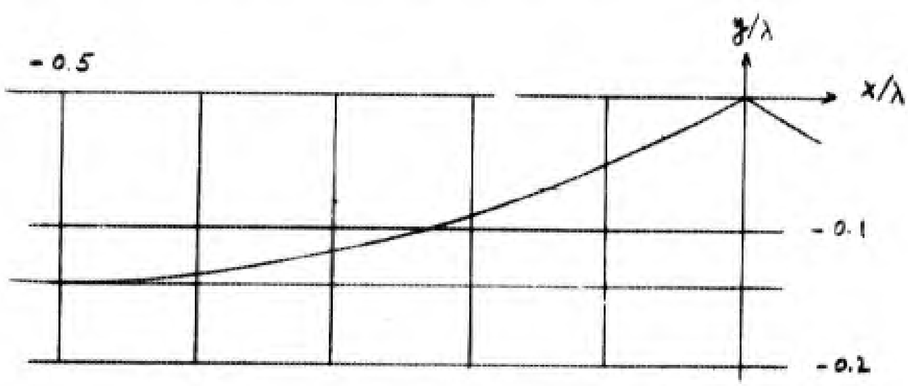


FIGURE 47

33β. Havelock's approximation for gravity waves.

In a paper already cited several times above Havelock [1919] extended Michell's method of construction of periodic waves of maximum amplitude, outlined in the preceding section, to one for construction of periodic waves of any allowable amplitude-length ratio. Up to the present, no one has proved the series involved to converge. However, as Havelock points out, the method has attractive theoretical features: the parameter describing the family of waves occurs in the form $e^{-\beta}$ where β varies from 0, corresponding to the highest wave, to ∞, corresponding to infinitesimal waves.

The method starts out exactly like Michell's up to equation (33.19) except that it is not assumed that $\psi = 0$ corresponds to the free surface. We recall that in Michell's analysis the constant-pressure condition did not enter completely until after (33.19), in particular, in (33.21). Havelock assumes instead that this condition is to be satisfied on some other streamline, $\psi = -\alpha$, which will then be taken to correspond to the free surface. The condition may still be written in the form (33.21) provided that one replaces $\psi = 0$ by $\psi = -\alpha$. For $\psi = -\alpha$ one may write

$$f' = c\sqrt[3]{2}\, e^{-\frac{1}{3} i\pi(\phi - i\alpha)/c\lambda} \left(i \sin \pi(\phi - i\alpha)/c\lambda\right)^{1/3} \sum_{n=0}^{\infty} \gamma_n e^{-i2n\pi\phi/c\lambda}, \tag{33.31}$$

where $\gamma_n = c_n e^{-2n\pi\alpha/c\lambda}$, the c_n being the same as those in (33.19). Havelock shows that one may express $\partial|f'|^4/\partial\phi$ in the following form

$$\frac{\partial}{\partial\phi}|f'|^4 = \frac{4}{3}\pi 2^{4/3}\frac{c^3}{\lambda}\sin\frac{\pi\phi}{c\lambda}\left[\sinh^2\frac{\pi\alpha}{c\lambda}+\sin^2\frac{\pi\phi}{c\lambda}\right]^{-1/3}\left[A_1\cos\frac{\pi\phi}{c\lambda}+A_3\cos\frac{3\pi\phi}{c\lambda}+\right]$$

(33.32)

and $\operatorname{Im} f'$ in the form

$$\operatorname{Im} f' = \frac{1}{3}c\, e^{-4\pi\alpha/3c\lambda}\sin\frac{\pi\phi}{c\lambda}\left[\sinh^2\frac{\pi\alpha}{c\lambda}+\sin^2\frac{\pi\phi}{c\lambda}\right]^{-1/3}\left[B_1\cos\frac{\pi\phi}{c\lambda}+B_3\cos\frac{3\pi\phi}{c\lambda}+\right].$$

(33.33)

Here the A_n's are rather complicated expressions in the γ's but also involve $\cosh \pi\alpha/c\lambda$ linearly; the B_n's are linear expressions in the γ_n with coefficients which are functions of $e^{-2\pi\alpha/c\lambda}$. Havelock finds complete expressions for the B_n's; for A_1, A_3, A_5, A_7 he finds the dependence upon the first few γ_n's. One must refer to the original for details, especially for the scheme for approximate solution for the γ_n's.

When $\alpha = 0$ the above analysis is precisely that for the highest wave. The numerical results of Havelock's computations for this case were given in the last section. He also computes $g\lambda/c^2$, γ_1, γ_2 (also γ_3 for the first) for two further cases: $e^{-2\pi\alpha/c\lambda} = 0.75$ and 0.3. The agreement with results computed by other methods, either those of section 27α or similar ones, is very close. However, to establish the validity of the method, one must prove convergence of the series $\Sigma|\gamma_n|$.

The relation of this method of approximation to Stokes' "second method" (see section 32γ) is also clarified by Havelock.

For this we refer to the original paper.

34. Explicit solutions

Although it is not in general possible to give an explicit exact solution to a particular problem of interest, it is possible to give various classes of exact solutions and then to determine the associated solid boundaries. This is sometimes referred to as an "inverse method". Several such methods for constructing exact solutions will be discussed in section 34 α. In addition, there is one periodic wave in infinitely deep fluid which satisfies the boundary conditions exactly, the Gerstner wave. This will be discussed in section 34 β. In section 34 γ we shall discuss briefly what may be called pseudo-exact solutions due to Davies and Packham. In these the exact boundary condition is replaced by a closely related one which allows exact solution. They derive their interest from the fact that they contain in one family waves ranging from the smallest amplitude-length ratio up to a counterpart of the Michell wave. Furthermore, the procedure also can be used for pseudo solitary and cnoidal waves. Section 34 δ will be devoted to an exact solution for pure capillary waves recently discovered by Crapper [1957].

34 α. Inverse methods

Sautreaux's method. Possibly the earliest method capable of generating a wide class of steady irrotational solutions is due to C. Sautreaux [1893, 1894, 1901]. It has been rediscovered several times subsequently, e.g., by Blasius [1910], Wilton [1913], Richardson [1920] and Lewy [1952]. F. Aimond [1929] has given a very comprehensive treatment of the method and of various

related ones. The method may be easily generalized to include an arbitrary impressed pressure distribution on the free surface (see the papers of Richardson or Aimond).

Let $z = x + iy$, $f = \phi + i\psi$, and take f as the independent variable. The free surface will be represented by $\psi = 0$. We further assume $q^2 > \varepsilon > 0$. In the constant-pressure condition on the surface, $\frac{1}{2}q^2 + g\eta$ = const., it will be convenient to take the position of the x-axis so that the constant is zero, and hence $\eta \leq 0$. This condition may then be expressed in terms of $z(f)$ as follows [cf. (32.56)]:

$$z'(\phi)\,\overline{z'(\phi)}\,[z(\phi) - \overline{z(\phi)}] = -ig \tag{34.1}$$

Define

$$\mu(f) = \tfrac{1}{2} i\,[z(f) - \overline{z(f)}]. \tag{34.2}$$

Then $-\mu(\phi) = y(\phi)$, the y-coordinate of the free surface. Hence, from (34.1)

$$\mu(\phi) = \frac{1}{2g}\,\frac{1}{z'(\phi)\,\overline{z'(\phi)}}. \tag{34.3}$$

From (34.2) and (34.3) one may now derive

$$2\,[g\mu(\phi)]^{-1} - 4\mu'^2(\phi) = [z'(\phi) + \overline{z'(\phi)}]^2 \tag{34.4}$$

Elimination of $\overline{z'}$ between (34.2) and (34.4) yields

$$z'(\phi) = -i\mu'(\phi) + \sqrt{(2g\mu)^{-1} - \mu'^2}, \tag{34.5}$$

where

$$\mu(\phi) > 0, \quad 2g\mu(\phi)\mu'^2(\phi) \leq 1. \tag{34.6}$$

But then, since z' is an analytic function of f, at least near $\psi = 0$,

$$z'(f) = -i\mu'(f) + \sqrt{(2g\mu)^{-1} - \mu'^2} \tag{34.7}$$

and

$$z(f) = -i\mu(f) + \int \sqrt{(2g\mu)^{-1} - \mu'^2}\, df. \tag{34.8}$$

One may now reverse the procedure, select an arbitrary analytic function $\mu(f)$ satisfying (34.6) and construct the function $z(f)$ by means of (34.8). The resulting function will describe a flow for which $z(\phi)$ is the free surface. If (34.6) is satisfied only for some range of f, then for the remaining range one must treat the streamline $\psi = 0$ as a solid boundary.

One can use the preceding result to construct a flow if the form of the free surface is given. Let the surface be given in the form $x = \xi(y)$ in a neighborhood of some point of the surface. Since $y(\phi) = -\mu(\phi)$ on the surface, we may define $\sigma(\mu) = \xi'(y)$ $= x'(\phi)/y'(\phi)$; σ is an analytic function of μ for real μ as follows from the theorem of Lewy and Gerber cited near the beginning of section 32. Hence, from (34.7),

$$\sigma(\mu) = -\left[(2g\mu)^{-1} - \mu'^2\right]^{1/2} / \mu'(\phi). \tag{34.9}$$

Solving for $1/\mu'$, one finds

$$\frac{d\phi}{d\mu} = \sqrt{2g\mu(1+\sigma^2)}. \tag{34.10}$$

Since μ is also an analytic function of ϕ, the same relation holds for $df/d\mu$ when μ is complex, and consequently

$$f = \int \sqrt{2g\mu(1+\sigma^2(\mu))}\, d\mu \tag{34.11}$$

It follows similarly from (34.8) and (34.9), first for real μ, then for complex μ that

$$z = -i\mu - \int \sigma(\mu)\, d\mu. \tag{34.12}$$

Equations (34.11) and (34.12) thus provide a relation between f and z determined by the form of $\sigma(\mu)$ for real μ.

Rudzki's method. Rudzki [1898] has given a different formula for deriving exact solutions. The derivation and statement of the formula below differ somewhat from Rudzki's, but the result is equivalent.

Let $z = z(f)$ and write

$$z' = \frac{1}{q} e^{i\theta}, \quad q = q(\phi,\psi), \quad \theta = \theta(\phi,\psi). \tag{34.13}$$

The free-surface condition may be expressed as follows, from (32.61) and the equation preceding it,

$$q^2 \frac{\partial q}{\partial \phi} = -g \sin\theta \quad \text{for } \psi = 0. \tag{34.14}$$

Hence

$$q = \left[-3g \int \sin\theta(\phi,0)\, d\phi\right]^{1/3} \quad \text{for } \psi = 0, \tag{34.15}$$

where the branch of the cube root is taken which is real for real numbers. Combining (34.15) with (34.13) gives

$$z'(\phi) = e^{i\theta(\phi,0)}\left[-3g \int \sin\theta(\phi,0)\, d\phi\right]^{-1/3}. \tag{34.16}$$

This relation must then hold also for $\psi \neq 0$, i.e.

$$z'(f) = e^{i\theta(f,0)}\left[-3g\int \sin\theta(f,0)\,df\right]^{-1/3}. \tag{34.17}$$

As in Sautreaux's method, we may now reverse the above procedure, take $\theta(f)$ as an arbitrary analytic function of f such that θ is real for f real, and construct $z'(f)$ from (34.17).

Richardson's method. From (34.17) one can derive immediately a formula due to Richardson [1920] for constructing exact solutions. Let $G'(f) = -\sin\theta(f)$. Then $e^{i\theta} = \sqrt{1-G'^2} - iG'$ and (34.17) becomes

$$z'(f) = [3gG(f)]^{-1/3}\left[-iG'(f) + \sqrt{1-G'^2}\right]. \tag{34.18}$$

Again, inversely, if $G(f)$ is any analytic function such that, for real f, G', $\sqrt{1-G'^2}$ and G are real, (34.18) gives a corresponding exact free-surface flow.

Examples: The largest collections of specific flows constructed by one of the preceding methods are in the paper of Richardson [1920] and a report of Vitousek [1954]. Several examples are given below.

(1) In (34.17) let $\theta(f) = \text{const.} = \alpha < 0$. Further, take the constant of integration as zero even though this results in a singularity in z' on the surface. One finds easily

$$f = \frac{2}{3}\sqrt{-2g\sin\alpha}\left(z e^{-i\alpha}\right)^{3/2}. \tag{34.19}$$

The free surface will consist of only the ray $z = re^{i\alpha}$ unless $\alpha = \pi/6$. However, the ray $z = re^{i(\alpha - \frac{2}{3}\pi)}$ is also a streamline,

but not one along which the pressure is constant unless $\alpha = -\pi/6$. Hence it must be taken as a solid boundary in general. The special case $\alpha = -\pi/6$ is just the flow (33.4) considered earlier and has a corner. One may, of course, take any other streamline $\psi = \psi_0 < 0$ as another solid boundary representing a bottom. The pressure remains positive everywhere only if $-\pi/6 < \alpha < 0$. This special family of flows was discussed by Weingarten [1904].

(2) If in (34.8) one takes $\mu(f) = f/c$ or in (34.18) takes $G(f) = \frac{2}{3}\sqrt{2g/c^3}\, f^{3/2}$, where c is some fixed velocity, one finds

$$c\,\zeta'(f) = -i + \sqrt{(2gf/c^3)^{-1} - 1} \qquad (34.20)$$

This yields a flow of the sort shown in Figure 48c, taken from Richardson. The internal solid boundary represents some streamline $\psi = \psi_0 < 0$. The free surface corresponds to the segment $\psi = 0$, $0 < \phi < c^3/2g$ in the f-plane.

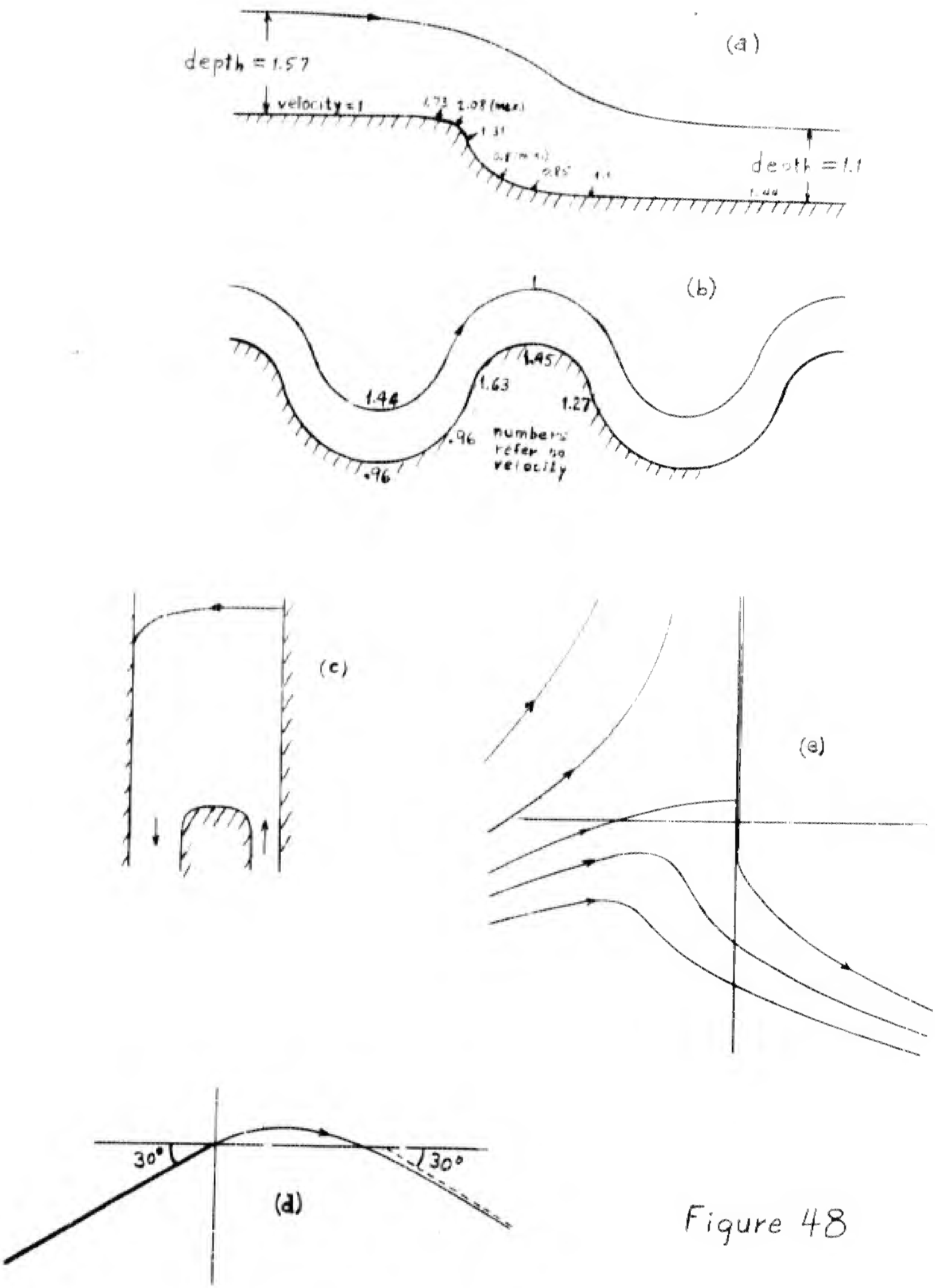
(3) Let c be some fixed velocity and let

$$G(f) = \frac{3c^3}{g}\left[B + \tanh \alpha \frac{g}{3c^3} f\right], \; B > 1, \; \alpha < 1,$$

in (34.18). Then

$$c\,\zeta'(f) = \left[B + \tanh \alpha \frac{g}{3c^3} f\right]^{-1/3} \left\{ -i\alpha \operatorname{sech}^2 \alpha \frac{g}{3c^3} f + \sqrt{1 - \alpha^2 \operatorname{sech}^4 \alpha \frac{g}{3c^3} f} \right\} \qquad (34.21)$$

Here $\psi = 0$ corresponds to the free surface. The choice of the bottom streamline is restricted by the necessity of avoiding having the singularity at $B = \tanh \frac{1}{3}\alpha g c^{-3} f$ within the fluid. Figure 48a, also from Richardson, shows a flow computed from (34.21) for $B = 2$, $\alpha = \frac{1}{2}$ and $c = 1$.



(a)
depth = 1.57
velocity = 1
1.73
2.08 (max)
.85
depth = 1.1
(b)
1.44
1.63
1.45
1.27
.96
.96
numbers refer to velocity
(c)
(e)
30°
30°
(d)

Figure 48

(4) Flow over a corrugated bottom has been investigated by both Richardson and Rudzki by essentially the same method. Following Richardson, we let

$$G(f) = \frac{3c^3}{g}\left[B - \cos\alpha\frac{g}{3c^3}f\right], \quad B > 1, \ \alpha < 1$$

Then

$$c\zeta'(f) = \left[B - \cos\alpha\frac{g}{3c^3}f\right]^{-1/3}\left\{-i\alpha\sin\alpha\frac{g}{3c^3}f + \sqrt{1-\alpha^2\cos^2\alpha\frac{g}{3c^3}f}\right\}. \tag{34.22}$$

Figure 48b shows a flow computed from this formula for $B=2, \alpha=0.9$.

(5) Flows similar to flows over a weir, under a sluice gate, through an opening, etc. have been considered by a number of the cited authors. Sautreaux [1901] applied his formula (34.8) with $\mu = (c^2/2g)e^{-2gi/c^3}$ to obtain a number of different flows of this nature. Figure 48d shows one of them. Lauck [1925] has also constructed such flows. Richardson obtained a flow through an opening by selecting

$$G(f) = \frac{3c^3}{g}\left[B - e^{gf/3c^3}\right].$$

Possibly the simplest such flow, studied by both Blasius and Vitousek, is obtained by taking $\mu = \sqrt{cf/g}$ in (34.8); this yields

$$\frac{g}{c^2}\zeta = -i\sqrt{\frac{gf}{c^3}} + \frac{1}{3}\left[2\sqrt{\frac{gf}{c^3}} - 1\right]^{3/2}. \tag{34.23}$$

The flow is shown in Figure 48e.

<u>Fritz John's method.</u> Fritz John [1953] has devised a method for constructing exact irrotational two-dimensional free-surface flows which may be time-dependent. Let $F(\zeta,t) = \Phi + i\Psi$ denote

the complex velocity potential. The flow of particles on the free surface, $y = \eta(x,t)$, will also be described in a Lagrangian system:

$$z = e(\alpha, t), \tag{34.24}$$

where α is a real number associated with a particular particle. Then

$$\frac{dz}{dt} = \frac{\partial e}{\partial t} = F'(x + i\eta(x,t), t), \tag{34.25}$$

where F' denotes the partial derivative with respect to z. The equations of motion (2.7), reduced to two dimensions and to motion along the free surface, given

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial \alpha} + \left(g + \frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial \alpha} = -\frac{1}{\rho}\frac{\partial p}{\partial \alpha} \tag{34.26}$$

Since p = const. on the surface, $\partial p/\partial \alpha = 0$ and (34.26) states that $\partial^2 z/\partial t^2 + ig$ is perpendicular to $\partial z/\partial \alpha$, or that

$$e_{tt} + ig = i\, r(\alpha, t)\, e_\alpha, \tag{34.27}$$

where $r(\alpha, t)$ is a real function. Thus $e(\alpha, t)$ must satisfy the parabolic partial differential equation (34.27).

If $e(\alpha, t)$ is a solution of (34.27) for some $r(\alpha, t)$ which is real for real α and if e and e_t are analytic functions of α and real for real α, then one may construct the velocity potential $F(z,t)$ for a free-boundary flow as follows. Actually, we shall construct F as a function of α and t, i.e. we shall construct a function G related to F by $G(\alpha,t) = F(e(\alpha,t), t)$. For real α it follows from (34.25) that

$$G_\alpha = F' \frac{\partial \zeta}{\partial \alpha} = \overline{e_t(\alpha,t)}\, e_\alpha(\alpha,t) = \overline{e_t(\bar{\alpha},t)}\, e_\alpha(\alpha,t). \tag{34.28}$$

One may now use the last expression in (34.28) to extend analytically G_α , and hence G from real to complex α's. By inverting $\zeta = e(\alpha,t)$, one may now construct $F(\zeta,t)$ (invertibility follows from (2.4) which implies $e_\alpha \bar{e}_\alpha = 1$).

It is possible to prescribe $\eta(x,t)$ and then construct the associated function $r(\alpha,t)$. For it follows from (34.26) with $y = \eta(x(\alpha,t),t)$ that

$$\frac{\partial^2 x}{\partial t^2} + \eta_x \left[\eta_x \frac{\partial^2 x}{\partial t^2} + \eta_{xx}\left(\frac{\partial x}{\partial t}\right)^2 + 2\eta_{xt}\frac{\partial x}{\partial t} + \eta_{tt} + g\right] = 0 \tag{34.29}$$

Any set of solutions $x(\alpha,t)$ depending upon a parameter α yields a function $e(\alpha,t)$ defined by

$$e(\alpha,t) = x(\alpha,t) + i\,\eta\,(x(\alpha,t),t). \tag{34.30}$$

The function $r(\alpha,t)$ for real α is given by

$$r(\alpha,t) = \frac{e_{tt} + i g}{i e_\alpha} = - \frac{x_{tt}}{\eta_x x_\alpha}, \tag{34.31}$$

where (34.29) has been used in obtaining the last expression.

We shall suppose now that the motion is steady and make the following special choice of Lagrangian parameter α . Select some fixed point ζ_0 of the surface $y = \eta(x)$ and for any particle on the surface let $-\alpha$ be the time at which the particle was at ζ_0. Since the motion is steady, all particles take the same amount

of time to travel from z_0 to any given point z and hence

$$e(\alpha,t) = e(0,t+\alpha) \equiv e(t+\alpha). \tag{34.32}$$

It then follows from (34.27) that also

$$r(\alpha,t) = r(\alpha+t). \tag{34.33}$$

Hence (34.22) becomes an ordinary differential equation in a single variable, say $\tau = t + \alpha$:

$$e''(\tau) - i r(\tau) e'(\tau) + i g = 0. \tag{34.34}$$

It follows next from (34.28) that $G(\alpha,t) = G(\alpha+t)$ and thus, if $e(\tau)$ is an analytic solution of (34.34), real for real τ,

$$G'(\tau) = \overline{e'(\bar{\tau})}\, e'(\tau) \tag{34.35}$$

In this case each choice of a function $r(\tau)$, real for real τ, results in a function $e(\tau)$ as a solution of (34.34), and then in a function $G(\tau)$ obtained by a quadrature of (34.35). One may invert $z = e(\alpha+t)$ and find F as a function of z as in the last paragraph or else regard

$$z = e(\tau) \quad , \quad F = G(\tau) \tag{34.36}$$

as parametric equations with complex parameter τ.

Several examples are considered by John, two of which are time-dependent. A simple and interesting steady flow is obtained by taking $r(\tau) = \nu$, a constant, in (34.34). Then (34.34) and (34.35), after setting the constants of integration equal to zero, yield

$$z = \frac{g}{\nu}\tau + A e^{i\nu\tau} \quad , \quad F = \left(\frac{g^2}{\nu^2} + \nu^2 A^2\right)\tau - 2\frac{g}{\nu} A \cos \nu\tau . \tag{34.38}$$

The free surface, obtained by taking τ real in the first formula, is a trochoidal curve without self-intersections if $A < g/v^2$; the wave length is $\lambda = 2\pi g/v^2$ and the amplitude is A. If $A < g/v^2$, then $|dF/d\mathfrak{z}| > 0$ and $A/\lambda < 1/2\pi$. However, $dF/d\mathfrak{z}$ can become infinite if $d\mathfrak{z}/d\tau = 0$. Such points occur at

$$\mathfrak{z} = (n+\tfrac{1}{4})\lambda + i\frac{\lambda}{2\pi}\left(1 - \log\frac{\lambda}{2\pi A}\right). \tag{34.39}$$

In order to avoid having them within the fluid, the bottom must be chosen as a streamline which passes above or through these points. Figure 49, taken from John's paper, shows several profiles and the associated streamlines through the branch points (34.39) computed for various values of the constant A when $\lambda = 2\pi$ (this is equivalent to graphing $2\pi\mathfrak{z}/\lambda$ for various values of $2\pi A/\lambda$). The surface profile and bottom come closer together as $A \to 1$ and draw further apart as $A \to 0$. For $A = 0.9$ they are so close that they cannot be conveniently separated in the figure; in such cases one may, of course

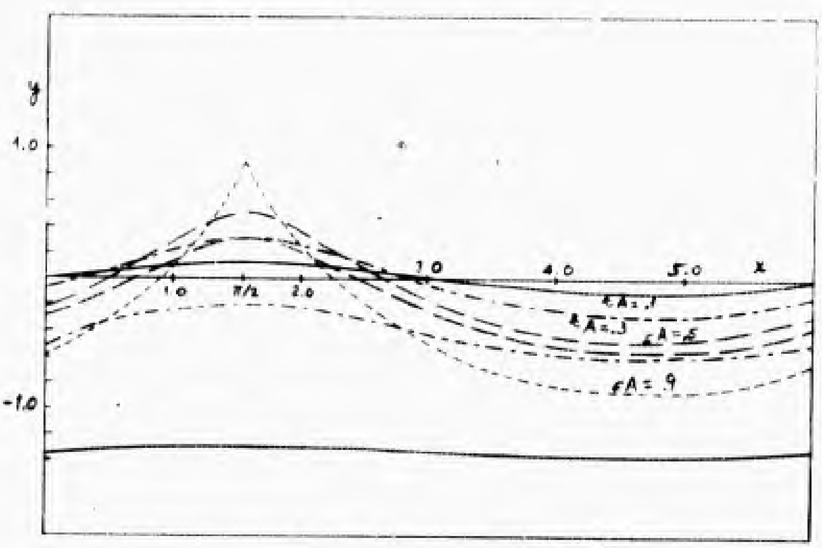


FIGURE 49

have reservations about the applicability of the perfect-fluid model.

The surface profile in this example is exactly the same as in the Gerstner wave treated in the next section. However, the Gerstner wave is defined for infinite depth and is not irrotational. The flow described above may also be obtained by Sautreaux's method. Vitousek [1954] has studied it by this procedure.

Methods of Villat and Poncin. At the end of section 32 brief mention was made of a pair of integral equations derived by Villat [1915] for the determination of flows over some given bottom profile and with the top profile also given upstream of some point. The method seems to be chiefly useful as an inverse method in which the free surface is given and the other profiles sought. Villat has worked out one case, but not in complete detail, where the top cover is missing and the bottom has a declivity.

Poncin [1932] has further elaborated Villat's method in the direction of starting with the fixed profile and finding the free profile. Actually, he does not really achieve this. Instead, he is able to construct a flow for a fixed profile of the same general behavior as the given one, but not identical with it. The method is applied to a number of interesting special cases, including flow over wavy bottoms and over bottoms with a declivity. The solutions are generally for large values of the velocity. The method and results do not lend themselves to a brief summary.

34β. Gerstner's wave

Gerstner's wave [1802] is apparently the first flow to have been discovered which satisfies exactly the condition of constant pressure on the surface, and is, in fact, one of the earliest papers on water-wave theory. It was subsequently rediscovered by Rankine [1863]. As will be shown below, the motion is not irrotational. This fact itself would not be enough to rule it out as a methematical model for real periodic waves. However, the direction of the vorticity is such that it is difficult to conceive of a scheme whereby such a wave could be generated in nature.

The motion is most easily described in Lagrangian coordinates. Each particle is associated with a pair of parameters (a,b), $b \le 0$. However, (a,b) does not represent a particular position of the particle at some time t_0, but instead a mean position. Hence, instead of (2.3) and (2.4) we need require insteady only that the determinant D of those formulas be independent of t. The motion is described by the equations

$$x = a + A e^{mb} \sin(ma + \sigma t), \quad y = b - A e^{mb} \cos(ma + \sigma t). \tag{34.40}$$

If $b = 0$ is taken as the free surface, the motion evidently represents a wave moving to the left with velocity $c = \sigma/m$, while the particles themselves describe in a counter-clockwise direction circular paths about the points (a,b) associated with the particles. The surface $b = 0$ is a trochoid and, in fact, each curve $b =$ const. < 0 is also a trochoid. In order that there shall be no self-intersections, one must have

$$A \le \frac{1}{m}. \tag{34.41}$$

In order to verify that the motion is kinematically possible, it is necessary to show, as noted above, only that the Jacobian $\partial(x,y)/\partial(a,b)$ is independent of t. An easy computation shows

$$\frac{\partial(x,y)}{\partial(a,b)} = 1 - m^2 A^2 e^{-2mb}, \tag{34.42}$$

so that the continuity condition is satisfied. Next one must show that the pressure is constant along the free surface. We shall, in fact, show more, namely, that it is constant along any line b = const. < 0, provided $\sigma^2 = gm$. To see this, introduce the equation (34.40) into the first of equations (2.7). A straightforward computation yields

$$A(gm - \sigma^2) e^{mb} \sin(ma + \sigma t) = -\frac{1}{\rho}\frac{\partial p}{\partial a} \tag{34.43}$$

If the pressure is constant along the surface, then $\partial p/\partial a = 0$. This can only hold if

$$\sigma^2 = gm \tag{34.44}$$

However, if $\sigma^2 = gm$, then $\partial p/\partial a$ for all b, so that each curve b = const. is an isobaric curve. Although we shall verify this fact directly, it now follows immediately from Burnside's theorem in section 32β that the motion cannot be irrotational. A direct computation of the vorticity vector is facilitated by noting that

$$\begin{aligned} u &= \frac{\partial x}{\partial t} = A\sigma e^{mb}\cos(ma+\sigma t) = -\sigma(y-b), \\ v &= \frac{\partial y}{\partial t} = A\sigma e^{mb}\sin(ma+\sigma t) = \sigma(x-a). \end{aligned} \tag{34.45}$$

Hence

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sigma\left(2 - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right), \tag{34.46}$$

The right-hand side of (34.46) may be computed from (34.40) by application of the rules of inversion for partial derivatives. The final result is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = - \frac{2\sigma m^2 A^2 e^{2mb}}{1 - m^2 A^2 e^{2mb}} ; \tag{34.47}$$

the negative sign indicates that the vorticity is directed oppositely to the orbital motion of the particles.

We shall omit a discussion of the construction of the curves b = const., streamlines in a coordinate system moving with the waves; it may be found in Lamb [1932, § 251], Milne-Thomson [1956, § 14.81] and in Kochin, Kibel' and Roze [1948, ch. 8, § 16] together with reproductions of Gerstner's original curves. It is, however, of interest to note that there is, according to (34.41), a "highest" wave of ratio $2A/\lambda = 1/\pi = 0.318$, a figure which may be compared with the value 0.142 for Michell's wave. The highest Gerstner wave has a cusp at the crests, a further indication that the motion cannot be irrotational.

The pressure distribution may be found by substituting (34.40) in the second equation of (2.7), using (34.44), and integrating. The result is

$$p = p_0 - \rho g b - \frac{1}{2}\rho\sigma^2 A\left(1 - e^{2mb}\right). \tag{34.48}$$

A computation of the potential and kinetic energies over a wave

length yields

$$T = V = \frac{1}{4} \lambda g \rho A^2 \left[1 - \frac{2\pi^2}{\lambda^2} R^2 \right]. \tag{34.49}$$

Finally, we note that nowhere in the preceding analysis have we made use of homogeneity of the fluid, i.e. the Gerstner wave also represents an exact solution for an arbitrary heterogeneous fluid (with ρ constant along streamlines). Moreover, Dubreil-Jacotin [1935] has shown that the Gerstner wave is unique in this respect.

Gerstner's wave is defined only for infinite depth. One may ask if a similar wave exists for finite depth. "Similar", in this context, will be taken to mean a periodic wave which satisfies exactly the constant-pressure condition on the free surface and for which the particle orbits are closed. Dubreil-Jacotin [1934] has proved the existence of such a wave and showed that it is unique when the period is fixed. However, this motion cannot be given explicitly except in the case of infinite depth. Methods of approximate computation of the wave have been given by Kravtchenko and Daubert [1957] and Gouyon [1958].

34γ. Pseudo-exact solutions

Although the solutions of this section are not really solutions satisfying the exact boundary conditions formulated earlier, they are exact solutions to a closely related problem, also with a nonlinear boundary condition. Their interest derives from the fact that it is possible to encompass within one explicit formula waves of all amplitudes up to a highest wave analogous to Michell's wave. The procedure also allows explicit construction

of solitary and cnoidal waves. It is possibly a misnomer to call these solutions pseudo-exact, for one may also interpret them as the first term in a certain series solution of the correctly posed problem. In this sense they are analogous to Havelock's approximation procedure described in section 33β. The work to be described has appeared in a series of papers by Davies [1951, 1952], Packham [1952] and Goody and Davies [1957].

The motion will be described in terms of the variables introduced in (32.85), $\omega = \theta + i\tau$. The alteration in the boundary condition consists in replacing (32.89) by

$$\frac{\partial \theta}{\partial \psi} = \lambda \frac{g}{c^3} e^{-3\tau} \sin 3\theta \quad \text{for } \psi = 0, \tag{34.50}$$

where λ is some fixed constant chosen so that $\lambda \sin 3\theta$ is a good approximation to $\sin \theta$. If one wishes to consider (34.50) as the first term in a series approximation to (32.89), one may expand $\sin \theta$ in a series in $\sin 3\theta$ and express (32.89) as

$$\frac{\partial \theta}{\partial \psi} = \frac{g}{c^3} e^{3\tau} \left[\frac{1}{3} \sin 3\theta + \frac{4}{81} \sin^3 3\theta + \dots \right]. \tag{34.51}$$

In this case (34.50) with $\lambda = 1/3$ represents the approximation obtained by keeping only the first term of (34.51). However, we shall not pursue the approximation procedure and refer to Davies [1951] for further information. It will be convenient to reformulate the boundary condition (34.50) as follows:

$$\operatorname{Im} \left\{ \frac{d\omega}{df} + \lambda \frac{g}{c^3} e^{i3\omega} \right\} = 0 \quad \text{for } \psi = 0, \tag{34.52}$$

or, after introducing the new variable $\chi(f) = e^{-i3\omega} = w^3/c^3$, as

$$\operatorname{Im}\left\{\frac{1}{\chi}\left(i\frac{d\chi}{df} + 3l\frac{g}{c^3}\right)\right\} = 0 \quad \text{for} \quad \psi = 0. \tag{34.53}$$

In order to proceed further, we must further specify the nature of the wave motion. Let us suppose the motion to be periodic with wave length λ and take the fluid to be infinitely deep. We again introduce the ζ-plane of (32.91) and take coordinates as in Figure 46. The expression in curly brackets in (34.53) is a regular analytic function of ζ inside the unit disc of the ζ-plane with vanishing imaginary part on the boundary, hence is a constant. Since, for $\zeta = 0$ (i.e. as $\psi \to -\infty$), $\chi = 1$ and $d\chi/df = 0$, the constant must be $3lg/c^3$. Thus χ must satisfy the differential equation

$$i\frac{d\chi}{df} - 3l\frac{g}{c^3}\chi = -3l\frac{g}{c^3}. \tag{34.54}$$

The solution is easily found to be

$$\chi = 1 + Ae^{-i3lgf/c^3} \tag{34.55}$$

Referring to Figure 46, we see that, if $f = 0$, $\chi = q_0^3/c^3$, where q_0 is the absolute velocity at a crest. Hence

$$A = \frac{q_0^3}{c^3} - 1. \tag{34.56}$$

Since θ must also vanish at $\phi = \pm\frac{1}{2}c\lambda$, i.e. the left-hand side of (34.55) must be real, we must also have $(3lg/c^3)\frac{1}{2}c\lambda = \pi$, or

$$c^2 = 3lg\lambda/2\pi. \tag{34.57}$$

Note that if $l = 1/3$, the relation between c^2 and λ is the same as in the infinitesimal-wave theory. The solution (34.55) may now be put into the following form:

$$\chi = \frac{w^3}{c^3} = 1 - \left(1 - \frac{q_0^3}{c^3}\right) e^{-i2\pi f/c\lambda} , \tag{34.58}$$

where $0 \le q_0 \le c$. If $q_0 = c$, then $w = c$ and the flow is uniform. If $q_0 = 0$, then

$$w^3 = c^3 \left[1 - e^{-i2\pi f/c\lambda} \right], \tag{34.59}$$

and near $f = 0, \pm c\lambda, \pm 2c\lambda, \cdots$ there is a corner in the wave profile with the two tangents making the same angle 120° as in Stokes' theorem (near $f = 0$, (33.5) gives $w^3 = i\frac{3}{2} g f$, (34.58) gives $w^3 = i 3 l g f$). Hence this wave corresponds to Michell's highest periodic wave. The ratio of amplitude to length of this wave may be computed from the following expression for the trough:

$$\frac{1}{2}\lambda - i a = \frac{1}{c}\int_0^{\frac{1}{2}c\lambda} \left[1 - e^{-i2\pi\phi/c\lambda}\right]^{-1/3} d\phi .$$

By expanding in a series and integrating term by term, one finds

$$\frac{a}{\lambda} = 0.127. \tag{34.60}$$

We recall that the value for Michell's wave was 0.142.

If the depth of fluid is finite, one must add the additional boundary condition, $Im\{\chi\} = 0$ for $\psi = -Q$, as well as for $\phi = 0$ and $\pm\frac{1}{2}\lambda c$. if the motion is to be periodic. The determination of χ now becomes too difficult to carry through briefly. How-

ever, an explicit solution is still possible and has been worked out by Davies [1952] and further investigated by Goody and Davies [1957]. Similarly, a "solitary wave" can be explicitly constructed which satisfies the boundary conditions $\operatorname{Im}\{\chi\}=0$ for $\psi=-Q$ and for $\phi=0$, $0>\psi\geq-Q$ and $\chi\to 1$ as $\phi\to\pm\infty$. This has been done by Packham [1952]. Either of these problems leads to the following differential-difference equation for $\chi(f)$:

$$\frac{1}{\chi(f+iQ)}\left[\chi'(f+iQ)-2i\frac{g}{c^3}i\right]+\frac{1}{\chi(f-iQ)}\left[\chi'(f-iQ)-2i\frac{g}{c^3}i\right]=0;$$

(34.61)

it may be established in a manner similar to that used in deriving (22.30) or (32.80).

34δ. Pure capillary waves

The first investigation of periodic progressive capillary waves satisfying the exact boundary condition is apparently due to N. A. Slëzkin [1937]. He formulated the boundary-value problem in the same manner as will be done below, reduced it to solution of a nonlinear integral equation analogous to Nekrasov's, and proved existence and uniqueness of a solution. However, he apparently did not observe that an explicit solution was possible for infinite depth of fluid. This was discovered by Crapper [1957], following a different and, in fact, more elementary method.

We shall consider the motion as a steady one in which the fluid moves to the right with velocity c as $y\to-\infty$. The existence of a complex velocity potential $f(z)=\phi+i\psi$ will be assumed and the free surface $y=\eta(x)$ will be taken to correspond to the streamline $\psi=0$ as usual. It will also be convenient to make

use of the variable $\omega = \theta + i\tau$ introduced in (32.85). If p_0 is atmospheric pressure, then from Bernoulli's integral

$$p + \frac{1}{2}\rho q^2 = p_0 + \frac{1}{2}\rho c^2 \tag{34.62}$$

(we recall that gravity is being neglected). The dynamical condition at the free surface [see (3.8) and (3.9)] is

$$p - p_0 = \frac{T}{R} = T\frac{\eta''}{[1 + \eta'^2]^{3/2}} \tag{34.63}$$

Before combining (34.62) and (34.63), we recall that the curvature of a streamline at any of its points is given by $d\theta/ds$ where s is arc length along the streamline. Hence, we may combine (34.62) and (34.63) to obtain the following boundary condition

$$\frac{1}{2}\rho(c^2 - q^2) = T\frac{d\theta}{ds} = T\frac{\partial\theta}{\partial\phi}\frac{d\phi}{ds} = Tq\frac{\partial\theta}{\partial\phi} \quad \text{for } \psi = 0, \tag{34.64}$$

or

$$\frac{\rho c}{2T}\left(\frac{c}{q} - \frac{q}{c}\right) = \frac{\partial\theta}{\partial\phi} \quad \text{for } \psi = 0. \tag{34.65}$$

Since $q = ce^{\tau}$ and since $\partial\theta/\partial\phi = \partial\tau/\partial\psi$ from the Cauchy-Riemann equations, (34.65) may be written

$$\frac{\partial\tau}{\partial\psi} = \frac{\rho c}{2T}\left(e^{-\tau} - e^{\tau}\right) = -\frac{\rho c}{T}\sinh\tau \quad \text{for } \psi = 0. \tag{34.66}$$

The problem is now to find a function $\omega(f)$ analytic for $\psi \le 0$, such that $\omega \to 0$ as $\psi \to -\infty$ and such that the imaginary part τ satisfies (34.66). However, since the boundary condition (34.66) involves only τ; unlike its analogue (32.89) for pure gravity

waves, it is possible to solve first for the harmonic function $\tau(\phi,\psi)$ and then to find θ later.

We assume that a solution can be found which satisfies

$$\frac{\partial\tau}{\partial\psi} = -h(\psi)\sinh\tau, \quad \tau(0) = \frac{\rho c}{T} \tag{34.67}$$

and proceed to verify the assumption. Integrating (34.67), we obtain

$$\log\tanh\tfrac{1}{2}\tau = -H(\psi) + G(\phi), \tag{34.68}$$

where $H'(\psi) = h(\psi)$ and $G(\phi)$ is an arbitrary function, or

$$\tau = \log\frac{e^{H} + e^{G}}{e^{H} - e^{G}} = \log\frac{X(\psi) + Y(\phi)}{X(\psi) - Y(\phi)}. \tag{34.69}$$

Since τ is a harmonic function of ϕ and ψ, Laplace's equation must be satisfied by (34.69). This yields an equation to be satisfied by X and Y in which the two variables can be separated. We shall not repeat the detailed analysis, which is typical of that occurring in separation-of-variables problems. The final result is that X and Y must satisfy

$$X'^2 = a_1 + a_2 X^2 + a_3 X^4,$$

$$Y'^2 = -a_1 - a_2 Y^2 - a_3 Y^4, \tag{34.70}$$

where a_1, a_2, a_3 are arbitrary constants. Crapper states that the full equations may be used to construct a solution for fluidof finite depth, but that it is sufficient to set $a_3 = 0$ for infinite depth (in view of Slëzkin's result, this is presumably also necessary). Since τ is real, we shall also take X and Y to be real.

If one does set $a_3 = 0$ and assumes $a_1 < 0$, $a_2 > 0$, the following give real solutions of (34.70):

$$X(\psi) = \sqrt{\frac{-a_1}{a_2}}\,\cosh(\sqrt{a_2}\,\psi + E),\quad Y(\phi) = \sqrt{\frac{-a_1}{a_2}}\,\cos(\sqrt{a_2}\,\phi + F), \tag{34.71}$$

where E and F are real constants. A glance at (34.69) shows that τ is independent of the choice of a_1. It will be convenient to let $a_2 = m^2/c^2$, where $m > 0$. One may determine E from (34.67), for

$$\frac{\rho c}{T} = H'(0) = \frac{d}{d\psi}\log X\Big|_{\psi=0} = \frac{m}{c}\tan E$$

or

$$e^{2E} = \frac{mT/\rho c^2 + 1}{mT/\rho c^2 - 1} \equiv B^{-2}. \tag{34.72}$$

Since E is to be real, we must evidently have

$$\frac{mT}{\rho c^2} \geq 1. \tag{34.73}$$

Since F adds only a real constant to ϕ we may select it at our convenience; we take $F = 0$. Substitution of (34.71) into (34.69) gives

$$\begin{aligned}\tau &= \log\frac{\cosh(m\psi/c + E) + \cos(m\phi/c)}{\cosh(m\psi/c + E) - \cos(m\phi/c)}\\ &= \log\frac{\cos(im\psi/c + iE) + \cos(m\phi/c)}{\cos(im\psi/c + iE) - \cos(m\phi/c)}\\ &= \log\left[\cot\tfrac{1}{2}(mf/c + iE)\cot\tfrac{1}{2}(m\bar{f}/c - iE)\right].\end{aligned} \tag{34.74}$$

The analytic function ω , which has τ as imaginary part and which approaches zero as $\psi \to -\infty$, is given by

$$\omega = i \log\left[-\cot^2 \tfrac{1}{2}(mf/c + iE)\right]. \tag{34.75}$$

We then have

$$\frac{df}{dz} = c e^{-i\omega} = -c \cot^2 \tfrac{1}{2}(mf/c + iE) = c \coth^2 \tfrac{1}{2}(imf/c - E). \tag{34.76}$$

From this one may solve for z in terms of f :

$$\begin{aligned} cz &= f - \frac{2c}{m} \tan \frac{1}{2}\left(\frac{mf}{c} + iE\right) + \text{const} \\ &= f - i\frac{4c}{m} \frac{1}{1 + e^{(imf/c - E)}} + \text{const} \\ &= f - i\frac{4c}{m} \frac{1}{1 + Be^{imf/c}} + i\frac{4c}{m}, \end{aligned} \tag{34.77}$$

where the constant has been chosen so as to make cz reduce to f when $B = 0$. It is evident that

$$z\left(f + \frac{2\pi c}{m}\right) = z(f) + \frac{2\pi}{m},$$

so that the streamlines are periodic in the x-direction with wave length $\lambda = 2\pi/m$.

The surface streamline is obtained by setting $\psi = 0$. After separation of real and imaginary parts in (34.77) the equation for the surface becomes:

$$\begin{aligned} x &= \frac{\varphi}{c} - \frac{4}{m} \frac{B \sin m\phi/c}{1 + B^2 + 2B\cos m\phi/c}, \\ y &= \frac{4}{m} - \frac{4}{m} \frac{1 + B\cos m\phi/c}{1 + B^2 + 2B\cos m\phi/c}, \end{aligned} \tag{34.78}$$

with ϕ serving as a parameter. There is a crest when $\phi = 0$ and a trough when $\phi = \pi c/m$. The difference in the values of y yields the following expression for the ratio of total amplitude to wave length:

$$\frac{a}{\lambda} = \frac{4}{\pi} \frac{B}{1-B^2} . \tag{34.79}$$

Equation (34.72) then provides a relation between A/λ and $mT/\rho c^2$, which we may write, for example, as

$$\begin{aligned} c &= \sqrt{\frac{Tm}{\rho}} \left(1 + \frac{1}{16} a^2 m^2\right)^{-1/4} \\ &= \sqrt{\frac{Tm}{\rho}} \left(1 - \frac{1}{64} a^2 m^2 + \cdots\right). \end{aligned} \tag{34.80}$$

If this formula is compared with (27.29), it should be kept in mind that a is here the total amplitude and that in (27.29) A is a length associated with the half amplitude. The formulas are consistant.

As a/λ increases, the surface profile becomes steeper and steeper near the troughs until the two sides finally touch. This occurs for $a/\lambda = 0.730$. A wave of these proportions may be considered as a "highest" capillary wave, an analogue of Michell's wave, although the nature of the limitation is different. Figure 50, reproduced from Crapper's paper, shows the profile of this wave together with other streamlines. It is a consequence of the form of the dependence in (34.77) that the other streamlines in Figure 50 may also serve as surface profiles for different values of a/λ, i.e. for different values of B. It is not surprizing, of course, that the profiles are similar to the middle three of

Figure 35.

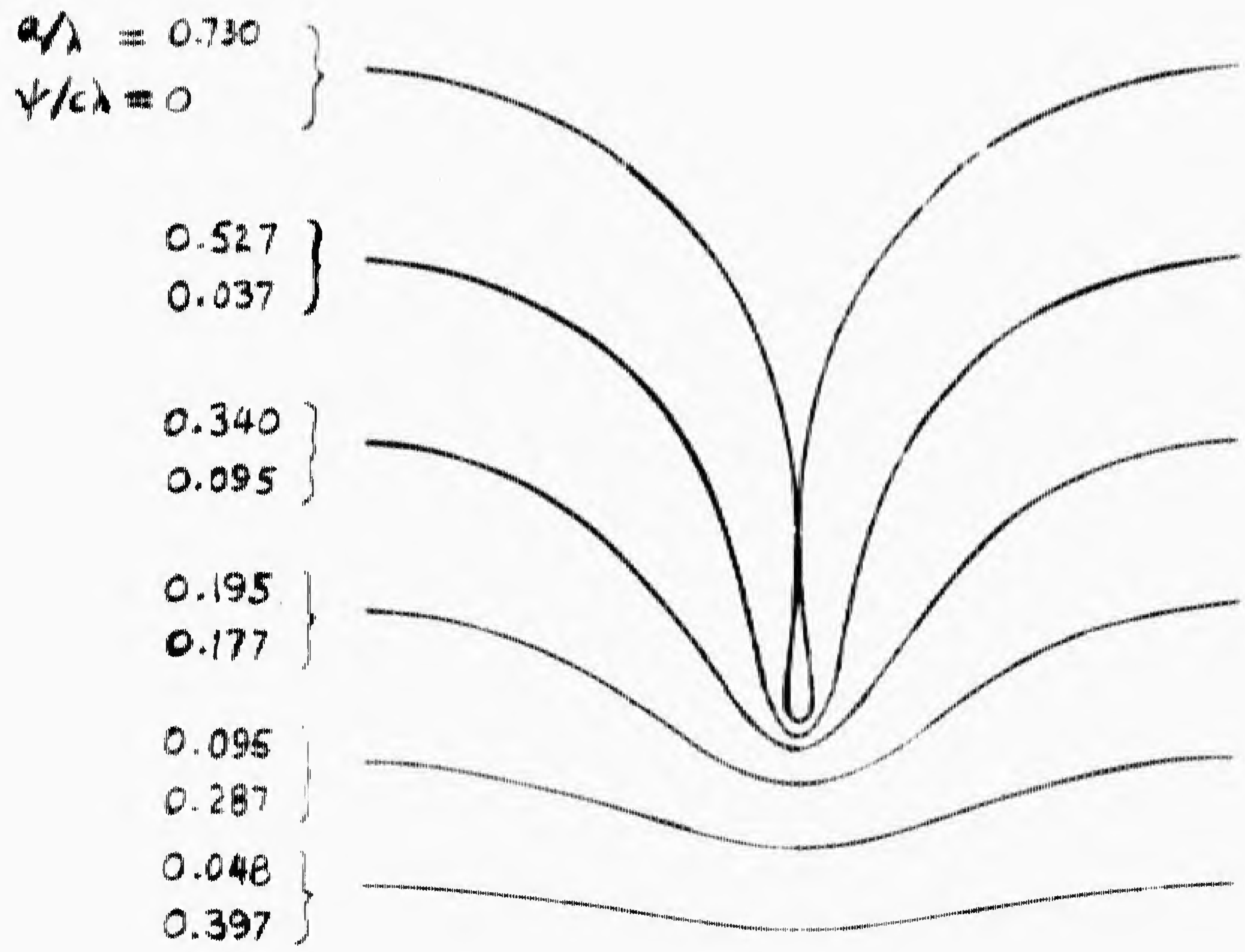


FIGURE 50

35. Existence theorems

In the various applications of the approximate theories of chapter D and E it is tacitly assumed that there is an exact solution which is being approximated. Without knowledge of conditions for existence and uniqueness of a solution to a particular problem, the status of an approximate solution is somewhat anomalous and one must rely upon comparison with experimental results for conviction concerning the correctness of the solution. However, such comparison is not a satisfactory criterion, for in the original formulation of a problem one will usually have already made a decision about the mathematical model of a fluid which will be used. Thus, if one has assumed a perfect fluid (as we usually have) and then made a further mathematical approximation in solving the problem at hand, the validity of this approximate solution must first be established before comparison of the predicted results with experimental measurements can be used as a criterion of the applicability of the fluid model. Without this additional knowledge, the comparison of approximate solutions with experimental results must be considered in some sense to be second best, even though valuable evidence may be provided by good agreement in a wide variety of situations.

Unfortunately, existence and uniqueness proofs in exact water-wave theory have generally been difficult to establish, and have usually been obtained for only rather restricted, although physically important, situations. Many of them are very recent and some rely upon methods of functional or topological analysis which cannot be briefly expounded. Although some proofs are so constructed that approximation methods are inherent in them,

others are only able to assert the existence of a solution with no indication of how to obtain it approximately. Proofs are still lacking for many relatively simple but important problems, for example, Michell's highest wave and standing water waves.

No attempt will be made to give an exposition of the mathematical methods which have been used in establishing the various existing theorems. Instead only a discursive account will be given of the nature and limitations of the known theorems.

35α. Irrotational waves - infinite depth

Proof of the existence of periodic waves of permanent type in infinitely deep water was first given by Nekrasov [1921, 1922] in a journal of very restricted distribution. Shortly thereafter Levi-Civita [1925] gave another proof along quite independent lines. Further proofs were given by Neumann [1929] and by Lichtenstein [1931], these being more closely related to Nekrasov's. A new treatment of Levi-Civita's proof, due to Littman and Nirenberg, is contained in Stokers' Water waves [1957, § 12.2]. Also, Nekrasov [1951] has recently published his researches in a more accessible form.

Nekrasov's method requires proving that there exists a solution $\theta(\gamma)$ to his nonlinear integral equation (32.104). His procedure, in brief, is to assume an expansion of $\theta(\gamma)$ in powers of the parameter $\mu' = \mu - 3 > 0$,

$$\theta(\gamma) = \mu'\theta_1 + \mu'^2\theta_2 + \cdots, \tag{35.1}$$

then to derive equations relating each θ_k to ones of lower index, and finally to show that the series converges. $\mu = 3$ is chosen

as a starting point because it is the first eigenvalue of the "linearized" equation (32.104), i.e. the one obtained by replacing the quotient containing $\sin\theta$ by simply $\mu\,\theta(\beta)$. This corresponds to the infinitesimal-wave theory. Proof of convergence requires that μ' be sufficiently small, and positive, but no estimate of radius of convergence is obtained. On the other hand, the method does allow computation of explicit approximate formulas for quantities of interest.

Levi-Civita also works with the variable ω, treating it as a function of the variable ζ introduced in (32.90). Hence his formulation of the problem is essentially the same as Nekrasov's, i.e. he is seeking a function $\omega(\zeta)$, regular in the disc $|\zeta| < 1$, vanishing at $\zeta = 0$ and satisfying (32.97) on $|\zeta| = 1$ and some further condition assuring that $|w/c - 1| < q < 1$. His procedure for finding such a function is to expand both ω and $k \equiv 1 - g\lambda/2\pi c^2$ in a power series in a parameter $\mu > 0$:

$$\omega = \sum_{n=1}^{\infty} \omega_n(\zeta)\mu^n , \qquad k = \sum_{n=1}^{\infty} k_n \mu^n , \tag{35.2}$$

where the functions $\omega_n(\zeta)$ and the constants k_n are to be determined by the boundary conditions. The first terms, $\omega_1 = -i\zeta$, $k_1 = 0$, correspond to infinitesimal waves, so that the parameter μ is essentially the amplitude/wave-length of this approximation. Levi-Civita establishes the convergence of the series (35.2) for sufficiently small values of μ. No estimate of a radius of convergence is given, but Hunt [1953] has stated that an examination and refinement of Levi-Civita's inequalities show that convergence is established for amplitude-wave length ratios

up to 1/98. The procedure lends itself to explicit computation of higher-order computations, and, in fact, he carries them out through $n = 5$. Levi-Civita further derives the interesting theorem that irrotational waves of permanent type must be symmetric about vertical lines through crest and trough. Nekrasov assumed this at the outset.

Neumann and Lichtenstein (the latter's approach is simpler) derive a coupled pair of nonlinear integral equations and put them into a form such that Schmidt's theory of nonlinear integral equations is applicable. Iterative methods of solution can be used to obtain approximate formulas.

35β. Irrotational waves - horizontal bottom

When the fluid is infinitely deep and the motion periodic, the only independent dimensionless parameter besides the amplitude-wave length ratio is $c^2/g\lambda$. When the fluid is bounded below by a horizontal plane at mean depth h, then a new parameter, say c^2/gh, must enter into any solution. However, other independent sets of parameters may be used, and, in particular, different choices of a perturbation parameter have led earlier to different approximate solutions for finite depth. Thus, in sections 14β and 27 one finds the first and higher approximations for periodic waves of permanent type when A/λ is taken as a perturbation parameter, whereas in section 31 one finds approximations to two further types of waves of permanent type, one of them periodic, corresponding to a different choice of parameter and a different method of approximation. In each of these cases there arises the question as to whether there exist waves of

permanent type satisfying the exact boundary conditions for which these waves may be considered approximations. In each case the answer is affirmative.

<u>Waves of small amplitude</u>. The first proof of the existence of periodic progressive waves in fluid of finite depth is due to Struik [1926]. His method of analysis is similar to Levi-Civita's for infinite depth. Existence of the desired wave is established for each value of $c^2/gh < 1$ and for each sufficiently small value of A/λ, where the bound on A/λ depends upon c^2/gh. Hunt [1953] has recently corrected some errors in the proof which did not invalidate it but which resulted in incorrect approximate formulas.

Nekrasov [1928, 1951] was also able to show that his integral equation for ζ, as modified for finite depth [see (32.104) and (32.106)], had a solution, thus providing an independent proof. As was the case for infinite depth, Nekrasov assumes that the waves are symmetric about verticals through trough and crest; Struik proves this. Krasnoselskii [1956] has recently applied topological methods of analysis to Nekrasov's equation and established not only existence of solutions for μ in the neighborhood of the eigenvalues of the linearized equation, but also their uniqueness and continuous dependence upon μ.

<u>Solitary and cnoidal waves</u>. Lavrent'ev [1943, 1947] was the first one to establish the existence of cnoidal and solitary waves. Cnoidal waves are not mentioned by him by name, but, in fact, their existence for sufficiently large wavelength is established along with that of the solitary wave, the latter being

obtained as a limiting case. The detailed exposition of the results [1947] is unfortunately both difficult of access and difficult to read, and relies upon earlier theorems of the authors. Although the perturbation parameter appears at first glance to be taken as $\varepsilon^2 = -1 - gh^3/Q^2$, which for the solitary wave would be in contradiction with (32.52), the quantity h is not really mean depth but a related quantity which varies with the wave length of the approximating periodic wave. Friedrichs and Hyers [1954] by a completely different procedure have established the existence of the solitary wave. Their perturbation parameter is essentially $\varepsilon^2 = 1 - gh^3/Q^2$ [actually it is $\alpha^2 = -\frac{1}{3}\log(gh^3/Q^2)$]. The point of departure is again the boundary condition (32.89) for the function ω. However, an integral equation is formulated, then altered by a change of variable $\hat{\varphi} = \alpha\varphi$, $\hat{\psi} = \psi$. The different rates of stretching correspond to those of section 10β . (Something like this also occurs in Lavrent'ev's proof, but is disguised in his theorems on conformal mapping of narrow strip-like regions.) An iterative procedure is used to prove existence of a solution for sufficiently small values of ε^2 .

Littman [1957] has used a method somewhat similar to that of Friedrichs and Hyers to establish the existence of cnoidal waves satisfying the exact boundary conditions. However, as a parameter he has used essentially h/λ , where h is the mean depth and λ the wave length. It is demonstrated that solutions exist for values of c^2/gh which are both greater and less than 1. Figure 51, modified slightly from one in Littman's paper shows in a qualitative fashion the relation between the dimensionless parameters. The dotted lines enclose values of the

parameters, again in a purely qualitative way, for which solutions have been demonstrated to exist. (Here k is the modulus of the elliptic functions and K is the complete elliptic integral of the first kind. h serves as a parameter in certain approximate formulas.)

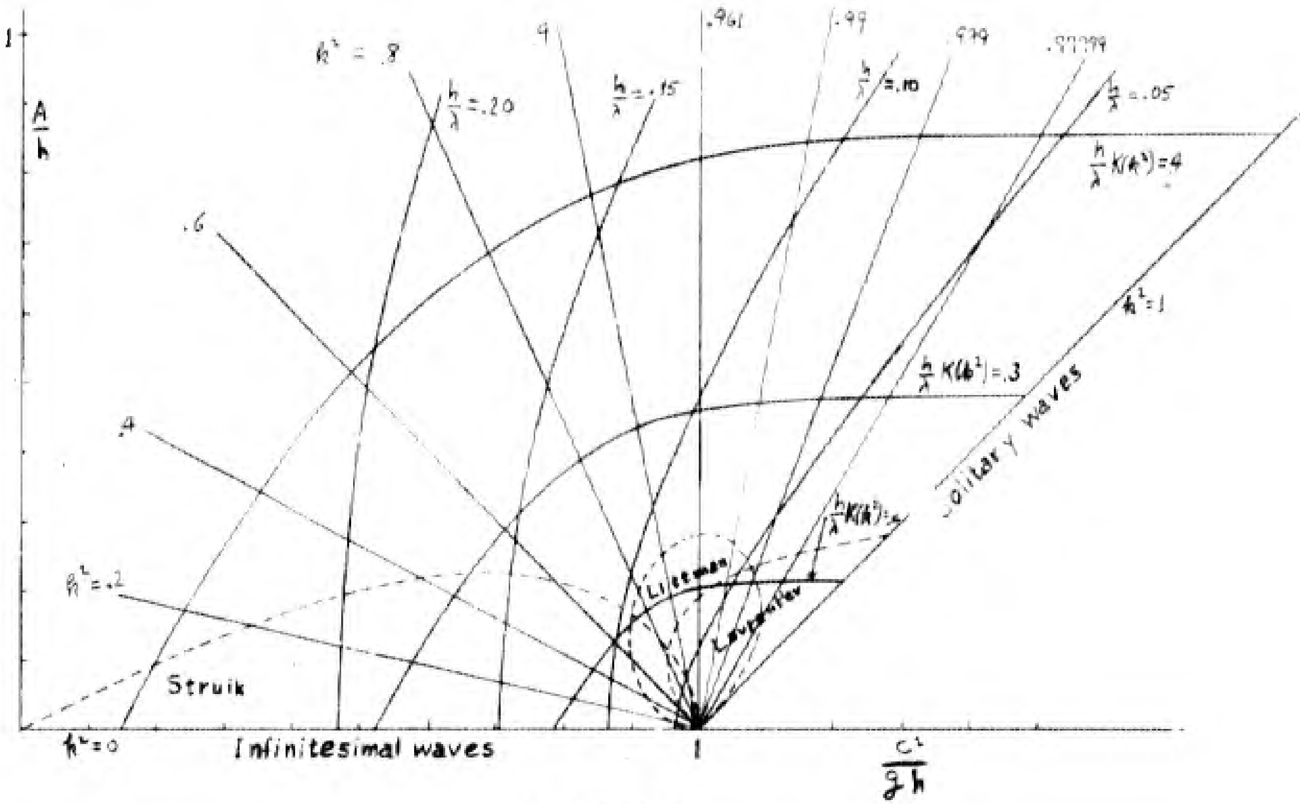


FIGURE 51

35γ. Irrotational waves - other configurations

Flow over a wavy bottom. In connection with the study of inverse methods in section 34α an explicit example of a steady flow over a wave-shaped bottom was exhibited. However, there the surface profile was given and the bottom profile calculated. The direct problem, in which the bottom profile and other flow data are given, has also been considered by several persons.

Lavrent'ev [1943] announced theorems concerning this problem, but did not include them in his later [1947] exposition. Gerber [1955] has given a comprehensive treatment of the "supercritical" case and has announced further results for the "subcritical" case [1956]. Let the bottom profile S be periodic and symmetric about vertical lines through the maxima and minima; let $\theta(s)$ be its intrinsic equation where s is arc length measured from a maximum and θ is the angle between the tangent and the x-direction. Let Q be the discharge rate for the fluid, and let q_0 be the velocity at a crest. In the first paper he considers flows in which the slope of the surface has the same sign as that of the bottom (we recall the two possible flows occurring in the linearized theory of section 20α). Gerber shows that there exists at least one solution of this type provided the following inequalities are satisfied in the interval between a maximum and the first minimum to the right:

$$\frac{gQ}{q_0^3} + \max|\theta| \leq \pi - \varepsilon_1, \qquad (35.3)$$

$$-\frac{1}{2}\pi + \varepsilon_2 \leq \theta(s) \leq 0,$$

where ε_1 and ε_2 are arbitrary small but positive quantities. If certain other inequalities, further limiting gQ/q_0^3, are satisfied, he is also able to prove uniqueness provided $\theta(s) \not\equiv 0$. In the second paper he announces that there exists at least one solution such that the profile has slope of opposite sign to that of the bottom if

$$\frac{gQ}{q_0^3} > (1+\varepsilon)\frac{\pi^3}{2} \qquad (35.4)$$

and $Q/L_0 g_0$ and $g_0 \Delta/Q$ are small enough; here L_0 is the arclength from a maximum to a minimum of S and Δ is the vertical distance. Gerber's methods are topological (Schauder-Leray theory) and do not yield effective methods of approximation.

Moiseev [1957] has also considered this problem. By a modification of the method used to derive Nekrasov's integral equation (32.104), he derives a pair of nonlinear integral equations to which the Lyapunov-Schmidt method is applicable. Let c be the average velocity defined by (7.5) for an allowable value of y (thus ϕ increases by $c\lambda$ over a wave length), and let Q be the discharge rate. Then Moiseev finds that there exists a sequence of velocities $c_1 > c_2 > \cdots > 0$ associated with the eigenvalues of a certain linear operator, such that, if $c \neq c_n$, there exists a unique flow provided the slope of the bottom is sufficiently small. Also, if $c > c_1$ or $c_{2n+1} < c < c_{2n}$ then the solution is such that the slopes of bottom and surface are of the same sign; if $c_{2n} < c < c_{2n+1}$, the slopes are of opposite sign.

Flow over a bottom with a declivity. Let the flow be from left to right and suppose the bottom profile to be asymptotic to horizontal lines as $x \to \pm\infty$, the one on the right being lower than that on the left. The discharge rate Q and velocity c at $x = -\infty$ should then be sufficient to determine the flow. The existence of a steady flow under these circumstances has been investigated by Haimovici [1935] and Gerber [1955]. The former derives a pair of nonlinear integral equations, similar to Nekrasov's, relating θ and τ of (32.86). An iterative method is used to prove the existence of a solution. Gerber makes use

again of the Schauder-Leray theory. The theorems established by each are very similar, but Gerber's is sharper. Let the bottom be given intrinsically by $\Theta(s)$, measured from some fixed point. Then a solution exists if

$$\frac{gQ}{c^3} < 1, \quad \max|\Theta(s)| + \frac{gQ}{c^3} < 1, \qquad (35.5)$$
$$|\Theta(s)| \leq A e^{-\alpha|s|}, \quad A, \alpha > 0.$$

The last condition assures a rapid approach to the horizontal asymptotes. The case of subcritical flow does not appear to have been treated in the published literature.

Motion past a submerged vortex. Ter-Krikorov [1958] has recently investigated steady flow past a submerged vortex of intensity Γ in a channel of depth h when the exact boundary conditions on the free surface are retained. If c is the velocity far upstream of the vortex, he proves existence and uniqueness of the flow provided that $c^2/gh > 1$ and Γ/ch is sufficiently small.

Interfacial waves. In section 14 δ we considered the linearized theory of waves at an interface between two perfect fluids of different densities, bounded above and below by horizontal planes. The question naturally arises as to whether one can establish the existence of such waves when the exact boundary conditions at the interface are observed. Kochin [1927] extended the methods of Levi-Civita and Struik to this problem and established the existence of (necessarily symmetric) inter-

facial waves of finite amplitude.

35 δ. Rotational waves

The explicit construction in section 34β of a periodic wave of permanent type which is rotational and the demonstrated existence of irrotational waves of this type which are of finite, if small, amplitude raises the question as to whether each of these waves is a special case of a more general type. This question has been discussed in a notable paper by Dubreil-Jacotin [1934] with results which include and generalize those of Levi-Civita and Struik. We givea only a bare indication of the results.

Let us suppose that a coordinate system has been chosen so that we may treat the wave motion as a steady flow to the right. Although we do not assume the motion to be irrotational, there will still exist a stream function $\psi(x,y)$ by virtue of the continuity equation. The vorticity of the flow will be given by $-\Delta\psi$, and since by a classical theorem the vorticity is constant along a streamline, the following equation must be satisfied by ψ:

$$\Delta\psi = f(\psi), \tag{35.6}$$

where $f(\psi)$ is some unspecified function. The condition on the free surface, $\psi = 0$, may still be derived from the special Bernoulli theorem [see equation (2.10")]

$$g\eta(x) + \tfrac{1}{2}[\psi_x^2 + \psi_y^2] = \text{const.} \tag{35.7}$$

For irrotational waves the function $f \equiv 0$, for Gerstner's wave

it is given by (34.47) after setting $\mathcal{b} = \psi$, $\sigma = cm$. The question which Dubreil-Jacotin asked is whether a wave of finite amplitude exists for any distribution of vorticity $f(\psi)$. In order to encompass both of the known finite waves into her results, she limits f to functions of the following sort:

$$f(\psi) = -\mu \frac{Q}{h} m^2 e^{2mh\psi/Q} F\left(e^{mh\psi/Q}\right), \quad -Q \le \psi \le 0, \tag{35.8}$$

where Q is discharge rate, h the me[illegible] depth, and the function $F(\rho)$ is bounded and satisfies a [illegible]uer condition in ρ; μ is a small parameter. If the depth is infinite, one must replace Q/h by c, the velocity at $\psi = -\infty$ (it is assumed that $\psi_y \to c$ as $\psi \to -\infty$).

Dubreil-Jacotin's theorem is as follows. For any $m = 2\pi/\lambda$, h and $f(\psi)$ satisfying (35.8) there exists a $\delta > 0$ such that for $\mu < \delta$ there exists a unique corresponding progressive wave of permanent type with vorticity distribution $f(\psi)$. The waves are also shown to be symmetric about vertical lines through crest or trough. She also demonstrates that among this class of waves for finite depth there is a unique analogue of the Gerstner wave, in the sense that the trajectories of individual particles are all closed. This wave has recently been investigated by Kravtchenko and Daubert [1957]. The development of means of calculating rotational waves has been the subject of a recent investigation by Gouyon [1958].

35ε. Waves in heterogeneous fluids - internal waves

It has been shown in section 32β that irrotational waves of

permanent type are not possible in a heterogeneous fluid, but that Gerstner's rotational wave still provides a solution for infinite depth. Dubreil-Jacotin [1935] has shown that this is the only periodic wave of permanent type in infinitely deep fluid having this property. In a later paper [1937] she returned to this topic and made use of the methods developed by her for rotational waves to investigate the existence theory for the two problems described below. The first problem, a natural generalization of one investigated by Kochin and mentioned at the end of section 35 γ, is the existence of periodic internal waves of permanent type in a heterogeneous fluid bounded both above and below by horizontal planes. In the second problem the upper surface is free.

The two problems may be formulated as follows. First we recall that in a steady flow of a heterogeneous fluid the density must be constant along streamlines. Hence, if ψ is the stream function, we may write $\rho = \rho(\psi)$. The equation analogous to (35.6) is now somewhat more complicated. It may be derived from (32.54) as follows. Apply the operators

$$\frac{\rho'}{\rho}\psi_y - \frac{\partial}{\partial y}, \quad \frac{\rho'}{\rho}\psi_x - \frac{\partial}{\partial x}$$

to the two equations of (32.54), respectively, and subtract. This yields

$$\frac{\rho'}{\rho}\frac{\partial(\zeta,\psi)}{\partial(x,y)} - \frac{\partial(\zeta,\psi)}{\partial(x,y)} = 0.$$

Since $\rho = \rho(\psi)$,

$$\frac{\rho'}{\rho}\frac{\partial(E,\psi)}{\partial(x,y)} = \frac{\partial(\rho' E/\rho,\psi)}{\partial(x,y)}$$

and hence

$$\frac{\partial(\rho' E/\rho - \zeta,\psi)}{\partial(x,y)} = 0$$

or, integrating and substituting $\zeta = -\Delta\psi$,

$$\Delta\psi + \frac{\rho'}{\rho}\left[gy + \tfrac{1}{2}(\psi_x^2+\psi_y^2)\right] = f_1(\psi) . \tag{35.9}$$

where $f_1(\psi)$ is an arbitrary function. This is the equation which ψ must satisfy. If $\psi = 0$ is the top streamline and $\psi = -Q$ the bottom streamline, then the boundary conditions are,

$$\psi = 0 \text{ for } y = 0 \quad , \quad \psi = -Q \text{ for } y = -h \tag{35.10}$$

for the first problem, and

$$\psi_x^2 + \psi_y^2 + 2gy = \text{const. for } \psi = 0,$$
$$\psi = -Q \text{ for } y = -h \tag{35.11}$$

for the second problem [cf. (32.60)]. The function $\rho(\psi)$ cannot be considered as an arbitrary given function in the same sense that $f_1(\psi)$ is arbitrary; it must be related to the density distribution when the fluid is at rest. Dubreil-Jacotin assumes that $\rho(\psi)$ is the same as the density at the mean level of the streamline ψ when the fluid is at rest.

In order to obtain results analogous to those of section

35δ, certain restrictions are placed upon the function $f_1(\psi)$ and the density distribution. Both problems are then reducible to integro-differential equations. In general there is no nontrivial solution. However, under certain conditions there are an infinite number of values of the parameter $\lambda g/2\pi c^2$ in the neighborhood of which there exist nontrivial symmetric waves of finite (but small) amplitude.

35ζ. Waves with surface tension

It has already been mentioned in section 34δ that Slëzkin [1935b, 1937] had derived an integral equation for the motion of pure capillary waves and had proved both existence and uniqueness of solution under certain circumstances. The explicit solution for this problem derived by Crapper supersedes in a sense these earlier results.

Sekerzh-Zenkovich [1956] has formulated the exact boundary-value problem for combined gravity and capillary waves in terms of the function ω of (32.86) and announced that a proof of existence for sufficiently small amplitude-to-wave length ratio can be carried out by Levi-Civita's method for pure gravity waves.